

The Hazard Potential: Introduction and Overview

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This is an expository article directed at reliability theorists, survival analysts, and others interested in looking at life history and event data. Here we introduce the notion of a hazard potential as an unknown resource that an item is endowed with at inception. The item fails when this resource becomes depleted. The cumulative hazard is a proxy for the amount of resource consumed, and the hazard function is a proxy for the rate at which this resource is consumed. With this conceptualization of the failure process, we are able to characterize accelerated, decelerated, and normal tests and are also able to provide a perspective on the cause of interdependent lifetimes. Specifically, we show that dependent life lengths are the result of dependent hazard potentials. Consequently, we are able to generate new families of multivariate life distributions using dependent hazard potentials as a seed. For an item that operates in a dynamic environment, we argue that its lifetime is the killing time of a continuously increasing stochastic process by a random barrier, and this barrier is the item's hazard potential. The killing time perspective enables us to see competing risks from a process standpoint and to propose a framework for the joint modeling of degradation or cumulative damage and its markers. The notion of the hazard potential generalizes to the multivariate case. This generalization enables us to replace a collection of dependent random variables by a collection of independent exponentially distributed random variables, each having a different time scale.

KEY WORDS: Competing-risk process; Degradation process; Dependence; Exchangeable lifetimes; Killing times; Lévy process; Marker; Multivariate failure models; Random killing; Reliability; Survival analysis.

1. INTRODUCTION AND OVERVIEW

1.1 Preliminaries: The Hazard Rate and the Hazard Potential

The mathematical theory of reliability, the statistical theory of life history or survival analysis, and the underlying premise of actuarial sciences are driven by a notion unique to them: the *hazard rate function* (see, e.g., Gjessing, Aalen, and Hjort 2003). The hazard rate function is both a theoretical and a descriptive tool that also plays a fundamental role in event history analysis. Specifically, there is a parallel between the hazard rate function and the *intensity function* of a nonhomogeneous Poisson process (see Grandell 1975), and also between the intensity function of a doubly stochastic Poisson process and the hazard rate function when the latter is viewed as a stochastic process (see Kebir 1991). There are two virtues of the hazard function: (a) an interpretive content, in the sense that the aging characteristics of single and one-of-a-kind items can be encapsulated by the shape of the hazard function, and (b) that under some regularity conditions (see Yashin and Arjas 1988; Singpurwalla and Wilson 1995), the hazard function uniquely determines a survival function. There are other scenarios in which (a) is also germane; these have been alluded to by Gjessing et al. (2003); some examples are an understanding of neuronal degeneration, the sleep-wake cycles of individuals, and the longevity of humans (see Gavrilov and Gavrilova 2001).

This is an expository article directed at reliability theorists, survival analysts, actuaries, and others interested in event history analysis. Our purpose here is to introduce a new notion, the *hazard potential* (HP) as a conceptual tool that provides a different way of looking at the stochastic behavior of lifetimes. The term “potential” refers to a feature parallel to that of *potential energy* in physics. The difference here is that we are alluding to an item's resistance to failure rather than its capacity for work. In Section 3 of this article we put forth the view that the HP can be interpreted as the (random) amount of an unknown resource

with which an item is endowed at inception, and that the item fails when this resource is depleted. Looking at lifetime in terms of a depleting resource can be more satisfying than one based on conditional probabilities, which is what the hazard function represents.

Besides providing an alternative platform for conceptualizing the process that leads to failure, and for processes that compete for failure, the HP has the following attractive features:

- It is inherently robust, in the sense that the HP of any and all items has an exponential (1) distribution on a suitably chosen time scale.
- It provides a context-free means for characterizing accelerated, decelerated, normal, and partially accelerated life tests.
- In the language of probabilistic causality (see Suppes 1970), it can be seen as either a *prima facie* or a genuine cause of dependence between lifetimes.
- It provides a vehicle for developing new families of univariate and multivariate survival functions by looking at the killing times of continuously increasing stochastic processes to random barriers.
- It offers a natural platform from which the abstract phenomenon of degradation (or damage accumulation) and its markers can be stochastically described.

The HP generalizes to the multivariate case. This generalization, when used in conjunction with the notion of a *hazard gradient* due to Marshall (1975a), enables us to represent a collection of dependent lifetimes in terms of a collection of independent exponential (1) random variables, each on a different time scale.

In light of the foregoing features, we may liken the HP to the notion of a *hidden parameter* in physics. Hidden parameters per se do not have a physical reality, but nonetheless are valuable because they provide explanations for observable phenomena.

1.2 Overview

This article is organized as follows. In Section 2 we introduce our notation and review some basic relationships. In Section 3

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we define the HP and interpret its nature from both physical and probabilistic standpoints. We also provide a way to formally distinguish between accelerated, decelerated, normal, and partially accelerated life tests from a context-free standpoint. The state of the art in accelerated testing seems vague when it comes to being specific about what a normal life test means; it treats this matter as a given. We conclude Section 3 by generalizing the HP to the nonexponential case through the notion of a *G-hazard potential*. In Section 4 we present several qualitative results pertaining to the claim that dependent HPs are a *prima facie* cause of dependent lifetimes, whereas a common HP is a genuine cause of dependence. Dependent HPs are a manifestation of commonalities in manufacturing or, in the context of biological units, a shared genetic makeup. In Section 5 we put the material of Section 4 to work by generating new families of dependent lifetimes using dependent HPs as a seed. In Section 6 we develop new families of survival functions for items destined to operate in random environments. The material here revolves around obtaining the distribution of the killing time of a continuously increasing stochastic process by a random barrier that is an item's HP. Although the approach of Section 6 is general, attention focuses only on the following processes: the running maxima of a Brownian motion, a Markov process with nonnegative increments, a family of nonnegative Lévy processes, and the integral of a geometric Brownian motion. The material of Sections 5 and 6 is not purely conceptual; it has the attractiveness of having a practical import. This can be seen as an argument in favor of looking at the HP as a useful tool. In Section 7 we explore the role of the HP in articulating the notion of competing-risk processes and casting the phenomenon of degradation and its markers in a manner that accords with that described in the engineering and materials science literature. We devote Section 8 to the multivariate case, which entails a relationship between the hazard gradient and what we introduce as a *conditional* HP. This connection allows us to replace a collection of dependent lifetimes by a collection of independent exponential (1) lifetimes, each indexed by a different time scale. In Section 9 we close the article by reemphasizing the point of view that the HP offers an alternative perspective for appreciating the failure process and that it is a useful conceptual tool for understanding the cause of interdependent lifetimes in engineering and biological systems. We close Section 9 by expressing our hope that the role of the HP could turn out to be as useful to reliability and survival analysis as the failure rate and the intensity functions.

2. NOTATION, TERMINOLOGY, AND PRELIMINARY RELATIONSHIPS

Let T denote the (unknown) time to failure of a unit that is scheduled to operate in some environment, labeled \mathcal{E} . Based on the characteristics of the unit, and on knowledge of how the unit interacts with \mathcal{E} (*vis-à-vis* T), one is able to subjectively specify $h(t)$, $t \geq 0$, the *hazard rate function* of $P(T > t)$, the *survival function* of T , assumed to be absolutely continuous. We interpret $h(t)$ through the relationship

$$h(t) dt \approx P(t \leq T \leq t + dt | T \geq t),$$

where the right side is a conditional probability. A formal definition of $h(t)$ can be found in the recent book of Aven and

Jensen (1999). We claim that the hazard function is a theoretical (or abstract) notion because, unlike lifetimes that can be directly observed, conditional probabilities are either subjectively specified or inferred from data.

Let $H(t) = \int_0^t h(u) du$; $H(t)$ is known as the *cumulative hazard* at time t . Observe that $H(t)$ is nondecreasing in t . But what does $H(t)$ mean? Whereas $h(t) dt$ can be given an intuitive import, $H(t)$ cannot! It is not the sum of conditional probabilities—because the conditioning event changes with t —and there is no law of probability that leads us to $H(t)$. Thus $H(t)$ does not have a probabilistic connotation. Yet $H(t)$ plays a key role in reliability and survival analysis, because of the *exponentiation formula* (see Barlow and Proschan 1975, p. 53), which says that with $H(t)$ specified,

$$P(T \geq t; H(t), t \geq 0) = e^{-H(t)}. \quad (1)$$

In the foregoing equation, plus those that follow, we introduce the convention that all quantities to the right of the semicolon are viewed as being specified. In contrast, all quantities to the right of the vertical slash are conditional, that is, if they are known.

Equation (1) relates the survival function $P(T \geq t)$ to $H(t)$; however, $H(t)$ lacks an interpretive content. Our interest in this article is motivated by the desire to interpret $H(t)$ in a manner that provides insight into the relationship of (1).

In the case of a one-of-a-kind item, $h(t) dt$ encapsulates an assessor's judgment about the inherent quality of an item and the environment in which it operates. By quality, we mean a resistance to failure-causing agents, such as crack growth, weakening of the immune system, and so on. Consequently, the hazard rate of an item of poor quality that operates in a benign environment could be smaller than that of a high-quality item that operates in a harsh environment. In effect, the quantity $h(t) dt$ encapsulates an assessor's subjective view of the manner in which an item and its environment interact. Thus, in principle, $h(t) dt$ does not have a physical reality.

Turning attention to the right side of (1), we note that $e^{-H(t)}$ is the survival function of an exponentially distributed random variable, say X , if its scale parameter is 1, evaluated at $H(t)$, that is,

$$P(T \geq t; H(t), t \geq 0) = e^{-H(t)} = P(X \geq H(t) | 1). \quad (2)$$

3. INTERPRETATION: THE NOTION OF A HAZARD POTENTIAL

Thus far, we have introduced three quantities, X , H , and T . Given any two of these, we can find the third using (2). But what insight can (2) provide about $H(t)$ and X ? We see two possibilities, one providing an indifference principle for reliability and survival analysis and the other having a physical connotation.

To appreciate the first, we see from (2) that, corresponding to every nonnegative random variable T having an absolutely continuous survival function $\bar{F}(t) = P(T \geq t)$, there exists a random variable X taking values $H(t)$, $0 \leq H(t) < \infty$, whose survival function is an exponential with a scale parameter of 1. The survival function of T is indexed by t , $t \geq 0$, whereas that of X is indexed by $H(t) = -\int_0^t d\bar{F}(u)/\bar{F}(u)$. We can summarize the foregoing in the following theorem.

Theorem 1. The lifetime of any and all items has an exponential (1) distribution on $H(t)$, the cumulative hazard, as the scale.

The essence of Theorem 1 has been noted by Cinlar and Ozekici (1987); it is stated here as a prelude to Theorem 5, which pertains to the multivariable case. In the context of point processes, Theorem 1 has a parallel with the result that any non-homogeneous Poisson process can be transformed by a change in clock time to a homogeneous Poisson process with rate one (see Kingman 1964). This parallel leads us to make precise the notions of accelerated and normal life tests in Section 3.1.

3.1 The Physical Connotation

To appreciate the physical connotation implied by (2), we note that because

$$P(T \leq t; H(t), t \geq 0) = P(X \leq H(t)|1),$$

we may claim that the time to failure T of an item coincides with the time at which the cumulative hazard $H(t)$ crosses a random threshold X , where X has an exponential (1) distribution (Fig. 1), that is, $T = H^{-1}(X)$.

The random threshold X , where $X = H(T)$, is defined as the HP of the item. Furthermore, because the exponential (1) distribution of X does not depend on \mathcal{E} , we may interpret X as an unknown resource with which the item is endowed at the time of its inception. With X considered a resource, $H(t)$ can be interpreted as the amount of resource consumed by time t . Consequently, the hazard rate, $h(t) = \frac{d}{dt}H(t)$, can be considered the rate at which the resource is consumed. With this alternative perspective on $H(t)$ and $h(t)$, we may view a *normal life-test* as one for which $H(t) = t$, a *uniformly accelerated (decelerated)* test as one for which $H(t) > (<) t$, and a *partially accelerated (decelerated)* test as one for which $H(t)$ crosses t from above (below). The qualifier accelerated (decelerated) signals a contraction (expansion) of the clock time from t to $H(t)$, and by shifting attention from the applied stress (which is what is normally done when discussing accelerated tests) to time, we achieve the context-free feature mentioned earlier. The concept

of looking at failure as the depletion of a resource dates back to a Soviet physicist Sedyakin (1966), who enunciated this viewpoint without a formal architecture.

It is useful to note that the exponential (1) random variable X has an entropy of 1, and also the lack of memory property if and only if $H(t) = t$. A change in clock time from t to $H(t)$ changes the entropy and destroys the memoryless property.

3.2 The G-Hazard Potential

There is a generalization of Theorem 1 such that the HP can be made to have a distribution other than an exponential (1). Specifically, suppose that G is some absolutely continuous distribution function with support $[0, \infty)$; let $W = G^{-1}$. Then it can be seen (Bagdonavicius and Nikulin 1999) that $Y \stackrel{\text{def}}{=} W(\bar{F}(T))$ has the survival function G , irrespective of \mathcal{E} . Consequently;

$$P(T \geq t) = P(W(\bar{F}(T)) \geq W(\bar{F}(t))) = P(Y \geq W(e^{-H(t)})),$$

so that

$$P(T \leq t) = P(Y \leq W(e^{-H(t)})). \tag{3}$$

Equation (3) implies that the item fails when $W(e^{-H(t)})$, exceeds a threshold Y , where Y has the distribution G . We refer to Y as the *G-hazard potential* and $W(e^{-H(t)})$ as the *G-resource used until time t*. Then we have, as a generalization of Theorem 1, the following result.

Theorem 2. The lifetime of any item can be made to have any absolutely continuous survival function G , provided that G is indexed by $G^{-1}(\exp(-H(t)))$.

As of now, Theorem 2 is mainly of an academic interest; it is given here for completeness.

4. HAZARD POTENTIALS AND DEPENDENT LIFETIMES

The aim of this section is to discuss the nature of dependence between lifetimes and offer a new perspective on the

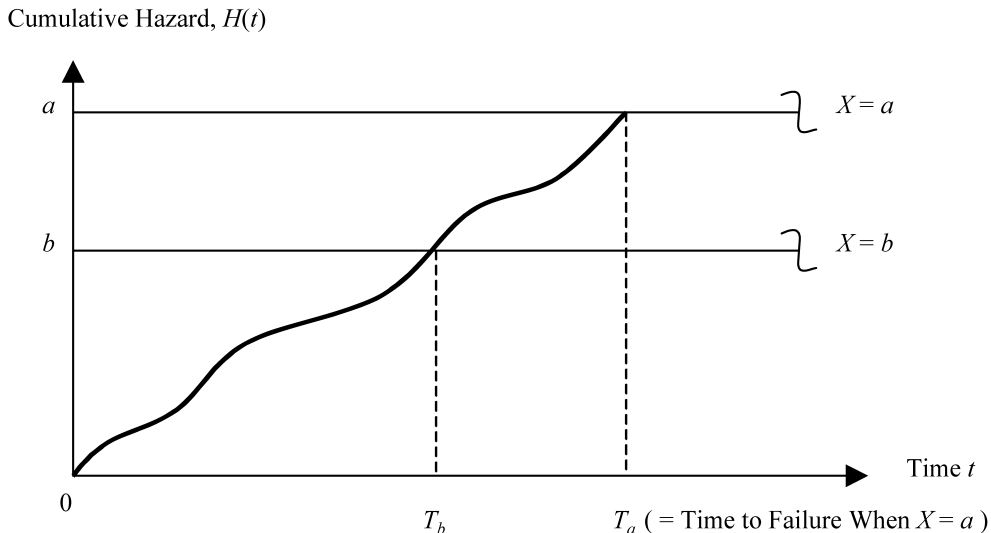


Figure 1. Relationship Between Cumulative Hazard, Threshold X , and Failure Time.

cause of interdependence. We argue that the HP offers a convenient platform for doing this. We view dependence and independence from a subjectivistic (de Finettian) viewpoint; that is, two events A and B are dependent if knowledge about B causes us to change our two-sided bets on A .

Because $H(t)$ encapsulates an assessor's view about the interaction between an item's quality and its environment, it is likely that two different items operating in a common environment will have different $H(t)$'s, say $H_1(t)$ and $H_2(t)$. Similarly, for a single item, changing its environment from \mathcal{E}_1 to \mathcal{E}_2 will change its cumulative hazard from $H_1(t)$ to $H_2(t)$ (Fig. 2).

Figure 2 suggests that the lifetimes T_1 and T_2 of two items having the same hazard potential will be dependent. Equivalently, the lifetimes T_1^* and T_2^* of a single item scheduled to operate in two environments, \mathcal{E}_1 and \mathcal{E}_2 , will also be dependent. However, from a subjectivistic perspective, the dependence will come into play only when one is able to specify $H_1(t)$ and $H_2(t)$, or a relationship between the two, when only one of them is known. This is because knowledge of, say, T_1 together with $H_1(t)$ will tell us something about the unknown X_1 , and if X_1 and X_2 are dependent, then knowledge of X_1 will enlighten us about X_2 . Consequently, X_2 together with $H_2(t)$ will help change our assessment of T_2 . To summarize, if the HPs X_1 and X_2 are dependent, then the lifetimes T_1 and T_2 will also be dependent, provided that $H_1(t)$ and $H_2(t)$ are known or a relationship between them is specified. In contrast, if X_1 and X_2 are independent, then so are T_1 and T_2 , irrespective of whether or not $H_1(t)$ and $H_2(t)$ are known. These assertions are summarized in the remarks that follow.

Remark 1. When $H_1(t)$ and $H_2(t)$, $t \geq 0$, are known, lifetimes T_1 and T_2 are independent if and only if their hazard potentials, X_1 and X_2 , are independent.

Proof. When X_1 and X_2 are independent,

$$P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2)) = P(X_1 \geq H_1(t_1)) \cdot P(X_2 \geq H_2(t_2)),$$

for any $H_1(t_1)$ and $H_2(t_2)$. Consequently,

$$\begin{aligned} P(T_1 \geq t_1, T_2 \geq t_2; H_1(t), H_2(t), t \geq 0) &= P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2)) \\ &= P(X_1 \geq H_1(t_1)) \cdot P(X_2 \geq H_2(t_2)) \\ &= P(T_1 \geq t_1; H_1(t), t \geq 0) \cdot P(T_2 \geq t_2; H_2(t), t \geq 0). \end{aligned}$$

Thus, knowing $H_1(t)$ and $H_2(t)$, T_1 and T_2 are independent, and similarly for the converse.

When $H_i(t)$, $i = 1, 2$ or both $i = 1$ and 2 , for $t \geq 0$ are not known, Remark 1 is weakened in the sense that only the "if" part holds. Specifically, T_1 and T_2 are independent even when X_1 and X_2 are dependent. The subjectivistic line of reasoning justifying this claim goes as follows.

Observing T_1 provides no insight about X_1 , because $H_1(t)$ is not known. Consequently, there also is no insight into X_2 or T_2 . Thus T_1 and T_2 are independent. Mathematically, without knowing $H_i(t)$, $i = 1, 2$, we are unable to relate $P(T_1 \geq t_1, T_2 \geq t_2)$ with the distribution of X_1 and X_2 . We summarize the foregoing in the following remark.

Remark 2. Lifetimes T_1 and T_2 are independent whenever $H_1(t)$ and (or) $H_2(t)$, $t \geq 0$, are not known.

As a consequence of Remarks 1 and 2, we may state the following theorem.

Theorem 3. Lifetimes T_1 and T_2 are dependent if and only if their hazard potentials X_1 and X_2 are dependent and if $H_1(t)$ and $H_2(t)$ are known.

Theorem 3 puts aside the often expressed view that the lifetimes of items sharing a common environment are necessarily dependent (see Marshall 1975b; Lindley and Singpurwalla 1986); that is, it is a common environment that causes dependence among lifetimes. Theorem 3 asserts that it is the commonalities in the HPs or identical HPs, both of which result in dependent HPs, that cause of interdependent lifetimes. Dependent HPs are a manifestation of similarities in design, manufacture, or genetic makeup. In the language of probabilistic

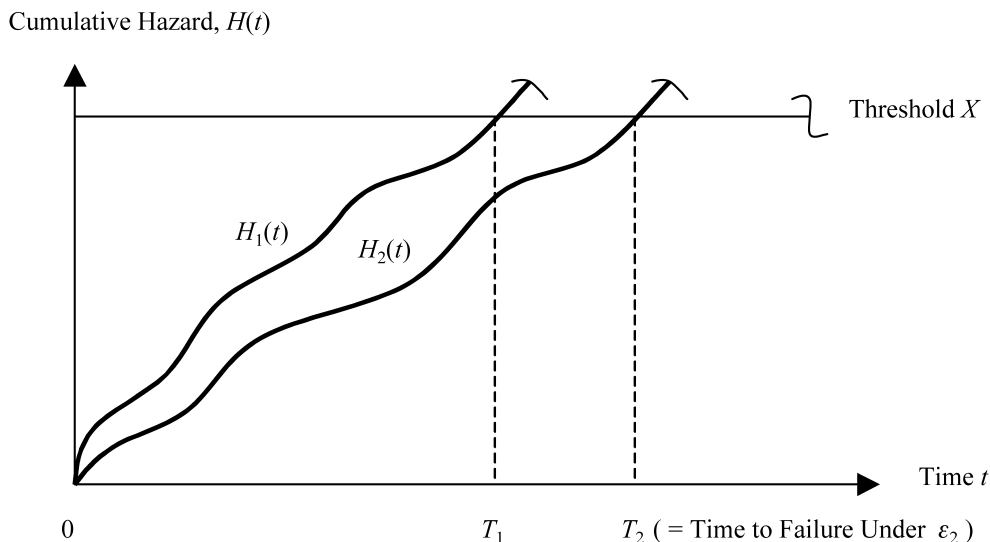


Figure 2. Effect of Changing Environment on Lifetimes.

causality of Suppes (1970), the common environment can be interpreted as a spurious cause of dependent lifetimes, whereas dependent (or identical) HPs are their *prima facie* (or genuine) cause.

The role of Theorem 3 is to generate new families of dependent lifetimes using multivariate distributions with exponential marginals as a seed; see Section 5. Remarks 1 and 2 pertain to the two extreme cases in which the $H_i(t)$'s, $i = 1, 2$, are either known or not. An intermediate case is one in which an $H_i(t)$, say $H_1(t)$, $t \geq 0$, is known and the other is not, except for the fact that $H_1(t) > H_2(t)$. For such scenarios, we have the following.

Remark 3. Suppose that $H_1(t) > (\leq) H_2(t)$ and that either $H_1(t)$ or $H_2(t)$, $t \geq 0$, is known; then X_1 and X_2 dependent implies that T_1 and T_2 are also dependent.

Proof. The proof is by contradiction. For this, suppose that X_1 and X_2 have the Bivariate Exponential Distribution (BVE) of Marshall and Olkin (1967); specifically, for λ_1, λ_2 , and $\lambda_{12} > 0$,

$$\begin{aligned} P(X_1 \geq x, X_2 \geq y) &= \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x, y)), \\ &= \exp(-(\lambda_1 + \lambda_{12})x - \lambda_2 y), \quad \text{if } x > y. \end{aligned}$$

The marginal distribution of X_i , $P(X_i \geq x) = \exp(-(\lambda_i + \lambda_{12})x)$, $i = 1, 2$. For the X_i 's to be dependent HPs, we need to have $(\lambda_i + \lambda_{12}) = 1$, for $i = 1, 2$, and $\lambda_{12} > 0$; this would imply that $\lambda_1 = \lambda_2 = \lambda$. Thus

$$P(X_1 \geq x, X_2 \geq y) = \exp(-(x + \lambda_2 y)).$$

If we set $x = H_1(t_1)$ and $y = H_2(t_2)$, for some $t_1, t_2 \geq 0$, then $x > y$ would imply that $H_2(t_2) = H_1(t_2) - \delta$, for some unknown $\delta > 0$. Consequently,

$$\begin{aligned} P(X_1 \geq x, X_2 \geq y) &= P(X_1 \geq H_1(t_1), X_2 \geq H_1(t_2) - \delta) \\ &= \exp(-(H_1(t_1) + \lambda_2(H_1(t_2) - \delta))). \end{aligned} \quad (4)$$

Given the foregoing, we need to show that T_1 and T_2 are dependent. Suppose that they are not; then

$$\begin{aligned} P(T_1 \geq t_1, T_2 \geq t_2; H_1(t_1), H_2(t_2), t_1, t_2 \geq 0) & \\ &= P(T_1 \geq t_1; H_1(t_1), t_1 \geq 0)P(T_2 \geq t_2; H_2(t_2), t_2 \geq 0) \\ &= P(X_1 \geq H_1(t_1))P(X_2 \geq H_2(t_2)) \\ &= \exp(-H_1(t_1))\exp(-(H_1(t_2) - \delta)) \\ &= P(X_1 \geq H_1(t_1), X_2 \geq H_1(t_2) - \delta), \end{aligned} \quad (5)$$

because the first term of (5) does not entail elements of the second term. Thus we have

$$\begin{aligned} P(X_1 \geq H_1(t_1), X_2 \geq H_1(t_2) - \delta) & \\ &= \exp(-(H_1(t_1) + H_1(t_2) - \delta)). \end{aligned} \quad (6)$$

Equation (6) agrees with (4) if $\lambda_2 = 1$. However, $\lambda_2 = 1$ implies that $\lambda_{12} = 0$, which contradicts the hypothesis that X_1 and X_2 are dependent. The proof when $H_1(t) \leq H_2(t)$ follows along similar lines.

A broader, but weaker version of Remark 1 pertains to the case where X_1 and X_2 are *exchangeable*. Here again, we require that $H_i(t)$, $i = 1, 2$, $t \geq 0$, be specified. We then have the following result.

Remark 4. If the hazard potentials X_1 and X_2 are exchangeable and if $H_1(t), H_2(t)$, $t \geq 0$, are known, then the lifetimes T_1 and T_2 are also exchangeable.

Proof. Let $x = H_1(t)$ and $y = H_2(t)$ for any $t_1, t_2 \geq 0$; then

$$P(X_1 \geq x, X_2 \geq y) = P(T_1 \geq t_1, T_2 \geq t_2; H_1(t_1), H_2(t_2)).$$

Similarly,

$$P(X_1 \geq y, X_2 \geq x) = P(T_1 \geq t_2, T_2 \geq t_1; H_1(t_1), H_2(t_2)).$$

Because the exchangeability of X_1 and X_2 implies that

$$P(X_1 \geq x, X_2 \geq y) = P(X_1 \geq y, X_2 \geq x),$$

the statement of the remark now follows.

5. GENERATING NEW FAMILIES OF DEPENDENT LIFETIMES

The aim of this section is to put Theorem 3 to work. Here we show how dependent HPs can be used to generate new families of multivariate distributions through multivariate distributions with unit exponentials as a seed. Of course, this is by no means the only way to generate multivariate distributions. For the purpose of illustration, we limit attention to the bivariate case and consider as seeds the bivariate exponentials of Marshall and Olkin (1967), Gumbel (1960), and Singpurwalla and Youngren (1993; henceforth S-Y), and a bivariate exponential induced by the copula of a bivariate Pareto distribution.

5.1 The Bivariate Exponential of Marshall and Olkin

Suppose that the HPs X_1 and X_2 have the BVE of Marshall and Olkin (1967), with λ_1, λ_2 , and λ_{12} as parameters. To ensure that the marginal distributions are unit exponentials, we need to have $\lambda_1 = \lambda_2 = \lambda$ and $\lambda + \lambda_{12} = 1$, with $\lambda_{12} > 0$; the latter inequality ensures dependence between X_1 and X_2 .

Let T_1 and T_2 be the lifetimes corresponding to X_1 and X_2 and the cumulative hazard functions $H_1(t_1)$ and $H_2(t_2)$. Then, because

$$\begin{aligned} P(T_1 \geq t_1, T_2 \geq t_2; \cdot) & \\ &= P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2); \lambda, \lambda_{12}) \\ &= \exp[-\lambda(H_1(t_1) + H_2(t_2)) - \lambda_{12} \max(H_1(t_1), H_2(t_2))], \end{aligned}$$

we can generate families of bivariate distributions for T_1 and T_2 , by assuming specific forms for $H_i(t)$, for $i = 1, 2$. In particular, if $H_i(t_i) = (\alpha_i t_i)^{\beta_i}$, $i = 1, 2$, then

$$\begin{aligned} P(T_1 \geq t_1, T_2 \geq t_2; \cdot) &= \exp[-\{\lambda[(\alpha_1 t_1)^{\beta_1} + (\alpha_2 t_2)^{\beta_2}] \\ &\quad + \lambda_{12} \max[(\alpha_1 t_1)^{\beta_1}, (\alpha_2 t_2)^{\beta_2}]\}], \end{aligned}$$

which is a bivariate Weibull of the Marshall–Olkin type.

If $H_i(t_i) = \alpha_i \ln(1 + \beta_i t_i)$, $i = 1, 2$, then

$$\begin{aligned} P(T_1 \geq t_1, T_2 \geq t_2; \cdot) & \\ &= \left(\frac{1}{1 + \beta_1 t_1}\right)^{\lambda \alpha_1} \left(\frac{1}{1 + \beta_2 t_2}\right)^{\lambda \alpha_2} \\ &\quad \times \min\left[\left(\frac{1}{1 + \beta_1 t_1}\right)^{\lambda_{12} \alpha_1}, \left(\frac{1}{1 + \beta_2 t_2}\right)^{\lambda_{12} \alpha_2}\right], \end{aligned}$$

which resembles the bivariate distribution of Muliere and Scarsini (see Kotz, Balakrishnan, and Johnson 2000, henceforth KBJ, pp. 408 and 595). This distribution is also known as the Marshall–Olkin–type Pareto distribution (see KBJ 2000, p. 612). Note that $H_i(t_i) = (\alpha_i t_i)^{\beta_i} [\alpha_i \ln(1 + \beta_i t_i)]$ corresponds to an increasing (decreasing) rate of consumption of the HP.

Continuing in the foregoing vein, if $H_i(t_i) = \alpha_i(e^{\beta_i t_i} - 1)$, $i = 1, 2$, then the induced distribution of T_1 and T_2 is given as

$$P(T_1 \geq t_1, T_2 \geq t_2; \cdot) = e^{(\alpha_1 + \alpha_2)\lambda} \cdot \exp\left[-\left\{\alpha_1 \lambda e^{\beta_1 t_1} + \alpha_2 \lambda e^{\beta_2 t_2} + \lambda_{12} \max(\alpha_1(e^{\beta_1 t_1} - 1), \alpha_2(e^{\beta_2 t_2} - 1))\right\}\right],$$

and if $H_i(t_i) = (1 - e^{-t_i})/(1 + e^{-t_i})$, $i = 1, 2$, the logistic function, then

$$P(T_1 \geq t_1, T_2 \geq t_2; \cdot) = \exp\left[-\left\{\lambda \left(\frac{1 - e^{-t_1}}{1 + e^{-t_1}} + \frac{1 - e^{-t_2}}{1 + e^{-t_2}}\right) + \lambda_{12} \max\left(\frac{1 - e^{-t_1}}{1 + e^{-t_1}}, \frac{1 - e^{-t_2}}{1 + e^{-t_2}}\right)\right\}\right].$$

Neither of these distributions is of a recognized form. The first form of $H_i(t_i)$ corresponds to an exponential rate of consumption of the HP, whereas the second corresponds to a rate of that which starts at $\frac{1}{2}$ at $t = 0$ and asymptotes to 1 as t becomes infinite.

5.2 The Bivariate Exponential of Gumbel

Following the notation of Section 5.1, suppose that for some parameter $0 \leq \theta \leq 1$,

$$P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2); \theta) = \exp[-H_1(t_1) - H_2(t_2) - \theta H_1(t_1)H_2(t_2)].$$

This is the bivariate exponential of Gumbel (1960), with marginals that are always unit exponentials. If $H_i(t_i) = (\alpha_i t_i)^{\beta_i}$, $i = 1, 2$, then the induced distribution of T_1 and T_2 is

$$P(T_1 \geq t_1, T_2 \geq t_2; \cdot) = \exp[-\{(\alpha_1 t_1)^{\beta_1} + (\alpha_2 t_2)^{\beta_2} + \theta(\alpha_1 t_1)^{\beta_1}(\alpha_2 t_2)^{\beta_2}\}];$$

we call this distribution the bivariate Weibull of the Gumbel type.

If $H_i(t_i) = \alpha_i \ln(1 + \beta_i t_i)$, $i = 1, 2$, then

$$P(T_1 \geq t_1, T_2 \geq t_2; \cdot) = \left(\frac{1}{1 + \beta_1 t_1}\right)^{\alpha_1} \left(\frac{1}{1 + \beta_2 t_2}\right)^{\alpha_2} \times \exp(-\theta \alpha_1 \alpha_2 \ln(1 + \beta_1 t_1) \ln(1 + \beta_2 t_2)),$$

which is a multivariate distribution with marginals that are a Pareto; we call this distribution a bivariate Pareto of the Gumbel type.

5.3 The Bivariate Exponential of S–Y

Here again, we follow the notation of Section 5.1 and suppose that for some parameter m ,

$$P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2); m) = \sqrt{\frac{1 - m \cdot \min(H_1(t_1), H_2(t_2)) + m \cdot \max(H_1(t_1), H_2(t_2))}{1 + m(H_1(t_1) + H_2(t_2))}} \times \sqrt{e^{-m \cdot \max(H_1(t_1), H_2(t_2))}}.$$

This distribution has unit exponential marginals if $m = 2$.

If we set $H_1(t_1) \geq H_2(t_2)$, then

$$P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2)) = \sqrt{\frac{1 - 2H_2(t_2) + 2H_1(t_1)}{1 + 2(H_1(t_1) + H_2(t_2))}} \exp(-2H_1(t_1)).$$

The multivariate distributions for T_1 and T_2 , when derived assuming that the $H_i(t_i)$ take any of the forms given in Section 5.1, are not of any recognizable type; they appear to be new. This is not surprising, because the bivariate exponential given earlier is also not of a well-recognized form.

5.4 Unit Exponentials Induced by Copulas

New families of multivariate distributions with unit exponentials can be created by the method of copulas and by invoking Sklar’s theorem in reverse (see, e.g., Nelson 1995). We can then use these multivariate exponentials as a seed for generating other families of multivariate distributions.

As an example of the foregoing, consider a bivariate Pareto distribution of the form

$$P(X_1 \geq x_1, X_2 \geq x_2; \cdot) = \left(\frac{b}{b + x_1 + x_2}\right)^{a+1};$$

its copula, for $u \geq 0$ and $v \leq 1$, is

$$C_a(u, v) = u + v - 1 + ((1 - u)^{-(a+1)} + (1 - v)^{-(a+1)} - 1)^{-(a+1)}.$$

If we set $u = 1 - \exp(-H_1(t_1))$ and $v = 1 - \exp(-H_2(t_2))$, then it can be seen (see Singpurwalla and Kong 2004) that

$$P(X_1 \geq H_1(t_1), X_2 \geq H_2(t_2); a) = \left(\exp\left(\frac{H_1(t_1)}{a + 1}\right) + \exp\left(\frac{H_2(t_2)}{a + 1}\right) - 1\right)^{a+1},$$

which is a bivariate distribution with unit exponentials as marginals. We may now choose any desired form for the $H_i(t_i)$, $i = 1, 2$, to produce new families of bivariate distributions of the form $P(T_1 \geq t_1, T_2 \geq t_2; \cdot)$.

6. CUMULATIVE HAZARD PROCESSES AND RANDOM KILLING

Our discussion thus far has been based on the premise that $H(t)$ is a deterministic function of t . This may be a reasonable first step. A more meaningful strategy is to assume that $H(t)$ is described by some nondecreasing and nonnegative stochastic process $\{H(t); t \geq 0\}$. There is some precedence for doing so in both the biostatistical and the reliability literature (see Singpurwalla 1995), although the motivation there is different

from what we give here. This is because we see $H(t)$ as a proxy for usage until time t , and conceptualizing usage as a random process is more natural than simply declaring that the cumulative hazard is a stochastic process. With $H(t)$ described as a stochastic process, the time to failure T will be the hitting time of $\{H(t); t \geq 0\}$ to a random barrier X , which is the HP of the item; see Figure 1. Put alternatively, the lifetime of an item corresponds to the killing time of $\{H(t); t \geq 0\}$ by a random threshold X . The notion that lifetimes correspond to hitting times of stochastic processes to some barrier was also explored in the pioneering work of Esary, Marshall, and Proschan (1973; henceforth EMP) and in the more recent works of Durham and Padgett (1997), Pettit and Young (1999), Yang and Klutke (2000), and Duchesne and Rosenthal (2003), the difference being that to these authors, the underlying stochastic process is an observable phenomenon such as degradation, aging, or cumulative damage. A consequence of the foregoing is that the results thus obtained pertain to specific scenarios. In contrast, the approach of considering any failure time as the hitting time of a process $\{H(t); t \geq 0\}$ to a random threshold X whose distribution is an exponential (1) provides a common architecture for developing classes of survival functions, with each class determined by the nature of the process. For example, when $\{H(t); t \geq 0\}$ is a positive nondecreasing Lévy process (special cases of which are the compound Poisson, the gamma, and the stable), a general result for the survival function is obtained. We discuss this and related matters in what follows.

6.1 The Hazard Rate and Cumulative Hazard Process

The purpose of this section is to obtain a result analogous to that of (2) when $H(t)$ is a stochastic process. To obtain an analog to the left side of (2), we proceed formally by considering a probability measure space (Ω, \mathcal{F}, P) on which all random variables and processes are defined.

Let $\{h(s); s \geq 0\}$ be a nonnegative and right-continuous stochastic process, and let T be a real-valued random variable denoting the lifetime of an item. For $t \geq 0$, we define the σ -algebras \mathcal{F}_t and \mathcal{F} as

$$\mathcal{F}_t = \sigma\{h(s); s \leq t\} \quad \text{and} \quad \mathcal{F} = \sigma\{h(s); s \geq 0\}.$$

Then $\{h(s); s \geq 0\}$ is defined as the *hazard rate process* of T , if, for $t > 0$,

$$P(T > t | \mathcal{F}) = \exp\left(-\int_0^t h(s) ds\right).$$

It now follows, from a result of Pitman and Speed (1973), that T is a *randomized stopping time*, so that

$$P(T > t | \mathcal{F}_t) = \exp\left(-\int_0^t h(s) ds\right), \quad t \geq 0.$$

Consequently,

$$P(T > t) = E\left[\exp\left(-\int_0^t h(s) ds\right)\right]$$

or

$$P(T > t) = E[\exp(-H(t))], \tag{7}$$

where $\{H(t); t \geq 0\}$ is the cumulative hazard process. Equation (7) is our analog of the left side of (2).

For an analog of the right side of (2), we assume that $\{H(t); t \geq 0\}$ is a nonnegative, nondecreasing stochastic process and consider the hitting time of this process to a random threshold X whose distribution is an exponential (1). Then, assuming independence of $H(t)$ and X ,

$$\begin{aligned} P(T > t) &= P(X > H(t)) = \int_0^\infty \exp(-y)H_t(dy) \\ &= E[\exp(-H(t))], \end{aligned} \tag{8}$$

where $H_t(\cdot)$ is the density of the distribution of $H(t)$. Thus an analog to the right side of (2), with $\{H(t); t \geq 0\}$ a stochastic process, is

$$P(T > t) = E[\exp(-H(t))]. \tag{9}$$

The right side of (9) is the Laplace transform of the process $\{H(t); t \geq 0\}$, which for the Lévy process has an explicit form, namely

$$E[\exp(-H(t))] = \exp\left[-t \int_0^\infty [1 - \exp(-y)]\nu(dy)\right], \tag{10}$$

where $\nu(dy)$ is the Lévy measure of the process and the integral term is the Laplace exponent of the Lévy process; complete the Lévy–Khinchin formula of Protter (1990). An attractive feature of the argument that leads to (8) is the straightforward manner in which it is developed. In contrast, the argument of (7) calls for some appreciation of randomized stopping rules associated with stochastic processes.

In what follows we consider several possible candidates for the process $\{H(t); t \geq 0\}$, starting with the simplest and moving to the more general. In most cases, explicit expressions for $P(T > t)$ are obtained; in others, computations and approximations may be needed.

The choice of which of the following processes to use depends on the application. Presumably, because $H(t)$ encapsulates the resource used until time t , the selection of a suitable process for $\{H(t); t \geq 0\}$ would depend on the pattern of use of the item.

6.2 Cumulative Hazard Processes and Their Survival Functions

The process $\{H(t); t \geq 0\}$ is required to be nonnegative, nondecreasing, and right-continuous. Thus our choice of candidate processes is limited. Clearly, the Brownian motion process, which has often been used to describe degradation and wear, must be eliminated. However, certain functionals of the Brownian motion, such as the running maxima, are viable candidates, and this is the first process considered.

6.2.1 The Maxima of Brownian Motion. Suppose that $\{W(t); t \geq 0\}$ is a standard Brownian motion process [i.e., $W(0) = 0$]; for any $t > 0$, $W(t)$ has a Gaussian distribution with mean 0 and variance t , and $\{W(t); t \geq 0\}$ has stationary independent increments. If we set

$$H(t) = \sup_{0 < s \leq t} \{W(s)\}, \quad t \geq 0,$$

then the process $\{H(t); t \geq 0\}$ will be continuous, nonnegative, and nondecreasing; this is called a *Brownian maximum process*.

It is well known that $T_x \stackrel{\text{def}}{=} \inf\{t \geq 0; W(t) \geq x\} = \inf\{t \geq 0; H(t) \geq x\}$, the time at which the process $\{W(t); t \geq 0\}$ first hits a barrier $x, x > 0$, has an inverse-Gaussian distribution (see Pettit and Young 1999). Consequently, the hitting time of the process $\{H(t); t \geq 0\}$ to x also has an inverse-Gaussian distribution, specifically,

$$P(T_x \leq t) = 2(1 - \Phi(x/\sqrt{t})),$$

where $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds$.

Because the time to failure of an item is the time at which the process $\{H(t); t \geq 0\}$ first hits the (HP) X , where X is exponential (1),

$$P(T \leq t) = \int_0^\infty 2(1 - \Phi(x/\sqrt{t}))e^{-x} dx = 1 - 2e^{t/2}\Phi(-\sqrt{t}),$$

so that

$$P(T > t) = 2e^{t/2}\Phi(-\sqrt{t}), \tag{11}$$

an expression that is easily evaluated.

6.2.2 The Compound Poisson Process. The compound Poisson process with an (arrival) rate λ and iid jumps $J_i, i = 1, 2, \dots$, with $P(J_i \leq \omega) = G(\omega)$ is another possible candidate for describing the process $\{H(t); t \geq 0\}$. This process increases only by jumps of size $J_i, i = 1, 2, \dots$. If we assume that the J_i 's are also independent of the HP X , then, given λ ,

$$P(T > t|\lambda) = \sum_{k=0}^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!} \int_0^\infty G^{(k)}(x)e^{-x} dx,$$

where $G^{(k)}(\cdot)$ is the k -fold convolution of $G(\cdot)$ with itself. The foregoing simplifies (see EMP 1973) as

$$P(T > t|\lambda) = \exp(-\lambda t), \tag{12}$$

for all $G(\cdot)$ with $G(x) = 0, x \leq 0$.

When $\lambda > 0$ is unknown, we may average out $P(T > t|\lambda)$ with respect to any distribution of λ . This would lead us to conclude that $P(T > t)$ has a hazard rate function $h(t)$ that is a decreasing function of $t \geq 0$. Thus items experiencing use of a resource described by a compound Poisson process will necessarily have lifetimes with a heavy-tailed distribution function. For example, if λ has a gamma distribution, then $P(T > t)$ will have a Pareto distribution, that is heavy-tailed.

6.2.3 A Special Markov Process. EMP (1973) considered a Markov process for $\{H(t); t \geq 0\}$ with a special feature that describes proneness to wear. Whereas their interpretation of $H(t)$ is unlike ours, their special feature is appropriate to our setup, specifically (a) $H(0) = 0$; (b) $H(t + \Delta) - H(t) \geq 0, \forall t, \Delta \geq 0$; and (c) $P(H(t + \Delta) - H(t) \leq u|H(t) = z) \downarrow z, t$. The practical import of (c) is that proneness to wear increases with usage. With the foregoing in place, EMP (1973) showed that for any barrier x , the hitting time of the process $\{H(t); t \geq 0\}$ has a distribution with a failure rate function $h(u)$ such that $\tilde{h}(t)$ increases in t , where

$$\tilde{h}(t) = \frac{1}{t} \int_0^t h(u) du. \tag{13}$$

Such distributions are said to have an increasing hazard rate average property. Because the barrier in our case is the HP X ,

where X has an exponential (1) distribution, we note that for the special Markov process for $\{H(t); t \geq 0\}$, the survival function $P(T > t)$ can be written as an exponential (1) mixture of distributions with the increasing hazard rate average property.

6.2.4 A Nonnegative Lévy Process. An omnibus way of describing $\{H(t); t \geq 0\}$ is through a nonnegative Lévy process, that is, a continuous process with stationary independent increments. Such processes are examples of Markov processes and include the compound Poisson, the gamma, and all stable processes as special cases. Furthermore, a Lévy process renews itself at stopping times and has a strong Markov property, and all of the nonnegative Lévy processes are limits of compound Poisson processes (see Protter 1990). Thus the process provides a convenient general platform for describing $\{H(t); t \geq 0\}$ and makes the result of (12) based on the compound Poisson process central. Besides the foregoing generalities, the main attraction of considering a Lévy process stems from the fact that its Laplace transform (given by the Lévy–Khinchin formula) takes a form identical to that of (10), namely

$$P(T > t) = \exp\left[-t \int_0^\infty (1 - \exp(-y))\nu(dy)\right], \tag{14}$$

where $\nu(dy)$, the Lévy measure, characterizes both the expected frequency and the size of the jumps (nonnegative in our case) in a Lévy process.

For the compound Poisson process of Section 6.2.2, $\nu(dy) = \lambda G(dy)$, and if G had a gamma distribution with scale $\alpha > 0$ and shape $\beta > 0$, then

$$\nu(dy) = \lambda \alpha^\beta y^{\beta-1} e^{-\alpha y} dy / \Gamma(\beta).$$

In the case of a gamma process [i.e., when for any $t \geq 0, H(t)$ has a gamma distribution with scale $\alpha > 0$, and shape βt],

$$\nu(dy) = (\beta e^{-\alpha y} / y) dy, \tag{15}$$

whereas when $\{H(t); t \geq 0\}$ is described by a stable process,

$$\nu(dy) = \frac{\alpha \beta}{\Gamma(1 - \beta)} y^{-(1+\beta)} dy \tag{16}$$

for parameters $\alpha > 0$ and $\beta \in (0, 1)$. Plugging (15) and (16) into (14) will give $P(T > t)$ for the special cases of the gamma and the stable process; also see (18) in the next section.

6.2.5 Continuous and Increasing Strong Markov Processes. One of the more striking results in stochastic processes theory pertains to continuous and increasing processes that have the strong Markov property. It has been shown that such processes have deterministic paths up to random killing. Essentially, this means that a continuous increasing strong Markov process is essentially deterministic. This result dates back to work of Blumenthal, Gettoor, and McKean (1962). Loosely speaking, if $\{H(t); t \geq 0\}$ is an increasing, continuous, strong Markov process with a state space of form $[a, b)$, then there exists a strictly increasing continuous function $k(\cdot)$ on the state space such that for all $t \geq 0, H(t) = k^{-1}[k(H(0)) + t]$; for specifics, see corollary 1 of Cinlar (1979). Thus the sample path of the $\{H(t); t \geq 0\}$ process is a deterministic function of the initial state of the process, namely $H(0) = 0$, and time t . Once the process $\{H(t); t \geq 0\}$ is considered (essentially) deterministic, obtaining the hitting time of $H(t)$ to a barrier is relatively

straightforward; it is also deterministic if the barrier is a known constant. Randomness of hitting times enters into the picture when the barrier is random, which is so in our case.

As an illustration of the foregoing, suppose that the process $\{H(t); t \geq 0\}$ is an increasing Lévy process. Recall that Lévy processes are continuous, have stationary independent increments, and thus are strong Markov. When this is the case, the function $k(\cdot)$ is such that for some $a \geq 0$,

$$H(t) = at + \int_0^\infty (1 - \exp(-ut))\nu(du), \quad (17)$$

where $\nu(du)$ is the Lévy measure of the process.

If we set $a = 0$ and assume that $\{H(t); t \geq 0\}$ is a gamma process, then $\nu(du)$ is given by (15), and the deterministic cumulative hazard function turns out to be

$$H(t) = \beta \log\left(\frac{\alpha + t}{\alpha}\right), \quad t \geq 0$$

(see Kebir 1991 for more details).

The unit fails when $H(t)$ gets killed by a threshold x ; that is, T_x , the time to failure for a fixed threshold x , is

$$T_x = \alpha(e^{x/\beta} - 1).$$

Averaging with respect to its exponential (1) distribution, we have

$$P(T \geq t) = \left(1 + \frac{t}{\alpha}\right)^{-\beta}, \quad (18)$$

which is a Pareto distribution. Note that the Pareto distribution also arises in the context of a compound Poisson process for $\{H(t); t \geq 0\}$ when the distribution of λ , the arrival rate, is assumed to be a gamma; see the discussion after (12).

To summarize, in practically all of the cases that we have considered so far, closed-form expressions for $P(T > t)$ are available. The sole exception is the special Markov process of Section 6.2.3, for which our result is merely qualitative. Our final case, considered next, pertains to an exponential functional of Brownian motion; here a closed-form result is not available. We chose this case because of its novelty and plausible applicability.

6.2.6 Integrated Geometric Brownian Motion Process. In Section 6.2.1 we considered the running maximum of a standard Brownian motion as a model for $\{H(t); t \geq 0\}$. Here we consider another functional. Specifically, let

$$H(t) = \int_0^t \exp(2W(s)) ds, \quad (19)$$

where $W(s)$ is a standard Brownian motion. We choose the scalar 2 for convenience; its role will become clear in the sequel. Observe that $\exp(2W(s))$ is always positive and that $H(t)$ is continuous and strictly increasing in t . Recall that a Brownian motion has continuous sample paths. Whereas $\sup_{0 < s \leq t} \{W(s); s \geq 0\}$ increases in t by steps, the $H(t)$ of (19) is a strictly increasing function of t . As stated earlier, Brownian motion has often been used to describe crack growth and degradation. The foregoing transformation of the process is necessitated by the requirement that $H(t)$ be nonnegative and non-decreasing. Our sense is that the $H(t)$ of (19) also could be a viable candidate for describing degradation and wear.

With the foregoing in place, we let

$$T_x = \inf\{t > 0: H(t) = x\}, \quad (20)$$

for some barrier $x \geq 0$; that is, T_x is the hitting (killing) time of the process $\{H(t); t \geq 0\}$ to a threshold x . Because $H(t)$ is continuous and increasing, we have that

$$P(T_x > t) = P(H(t) < x). \quad (21)$$

To evaluate the right side of the foregoing, we need to know the density of $H(t)$ for a fixed value of t . For convenience, we denote $H(t)$ by H_t , and, following the notation of Yor (1992), note that

$$\begin{aligned} \frac{P(H_t \in dv)}{dv} &= \sqrt{\frac{2\pi}{v}} \int_0^\infty \exp\left(-\frac{y^2}{2t} + \frac{v}{2} \cosh^2 y\right) \\ &\quad \times \sinh y \sin\left(\frac{\pi y}{t}\right) (1 - \Phi(\sqrt{v} \cosh y)) dy, \end{aligned}$$

where $\Phi(u)$ is as defined in Section 6.2.1. Consequently,

$$P(T_x > t) = \int_0^x P(H_t \in dv),$$

from which it follows that

$$P(T > t) = \int_0^\infty P(T_x > t) e^{-x} dx, \quad (22)$$

$$= \int_0^\infty \int_0^\infty P(H_t \in dv) e^{-x} dx, \quad (23)$$

with $P(H_t \in dv)$ as given earlier.

7. COMPETING-RISK AND DEGRADATION PROCESSES

7.1 Competing Risks and Competing-Risk Processes

Loosely speaking, the term “competing risks” connotes competing causes of failure, and interest centers on the cause of failure and/or the time to failure given that there are several agents competing for an item’s lifetime. The issue can be quite complex because the causes do not operate in isolation of one another, it often being the case that one causeacerbates the effect of the other. Traditionally, the model used for encapsulating the scenario of failure under competing risks is the reliability of a series system with independent (or dependent) component lifetimes, the latter representing the causes of system failure. In what follows, we shift focus from independent or dependent lifetimes to independent or dependent HPs to develop a framework that could provide a more realistic description of the competing-risk phenomenon. Accordingly, let T_i denote the time to failure of the i th component of a series system of k components, $i = 1, \dots, k$, and T the time to failure of a system. Then

$$P(T \geq t) = P(H_1(t) \leq X_1, \dots, H_k(t) \leq X_k),$$

where $H_i(t)$ is the cumulative hazard (or risk) experienced by the i th component and X_i is its HP. If the HPs are assumed to be independent, then

$$P(T \geq t) = \exp[-(H_1(t) + \dots + H_k(t))], \quad (24)$$

suggesting an additivity of the cumulative hazards (or risks). If the HPs are assumed to be dependent, then the nature of dependence would dictate the form taken by $P(T \geq t)$; see Section 5.

In either case, our expression for $P(T \geq t)$ would be the same as what we would obtain assuming the dependence or independence of the lifetimes T_i . Thus it would appear that little, if any, gain has been achieved by shifting focus from the T_i 's to the X_i 's. But there is another way to look at (24), a way that paves the path for obtaining another expression for the survival function of an item experiencing multiple risks.

Observe that (24) is also the survival function of a single item that has a cumulative hazard of $H(t) \stackrel{\text{def}}{=} \sum_{i=1}^k H_i(t)$, at time t . But when this is the case, how can we interpret each $H_i(t)$? More generally, in the case of a single item with a cumulative hazard of $H(t)$, can there be a meaningful decomposition of $H(t)$, and, if so, can it be additive? Moreover, which of the two perspectives more accurately reflects the competing-risk phenomenon?

One possible strategy for addressing these questions is to see each $H_i(t)$, $i = 1, \dots, k$, as the consequence of a covariate and to suppose that if the item were to experience covariate i alone, then its time to failure would coincide with the item at which $H_i(t)$ crossed its hazard potential X . With the item simultaneously experiencing k covariates, its survival function would be

$$\begin{aligned} P(T \geq t) &= P(H_1(t) \leq X, \dots, H_k(t) \leq X) \\ &= P(X \geq \max\{H_1(t), \dots, H_k(t)\}) \\ &= \exp(-\max\{H_1(t), \dots, H_k(t)\}). \end{aligned} \quad (25)$$

Clearly, under the scenario of an item simultaneously experiencing k causes of failure (risks), the decomposition of $H(t)$ is not additive.

Whereas (25) could be new to the literature on competing risks, it is worth noting that the two scenarios discussed earlier—the traditional one involving a series system that leads to (24) and the one pertaining to the single item that leads to (25)—are related because considering a single HP X is tantamount to considering k HPs that are totally (and positively) dependent on one another. This leads to the following result.

Theorem 4. The survival function under any series system model for competing risks with positively dependent hazard potentials is bounded as

$$\begin{aligned} \exp\left(-\sum_{i=1}^k H_i(t)\right) &\leq P(T \geq t) \\ &\leq \exp(-\max\{H_1(t), \dots, H_k(t)\}). \end{aligned}$$

This theorem shows that the two perspectives on competing-risk modeling can be reconciled through the notion of independent and dependent hazard potentials, with the left side of the inequality reflecting the former and the right side reflecting the latter.

7.1.1 Dependent Competing Risks and Competing Risk Processes. In our discussion thus far, the $H_i(t)$'s have been assumed known and specified. Consequently, the matter of independent or dependent competing risks was not germane; dependence and independence were embodied in the context of HPs. But the prevailing view of what constitutes dependent competing risks entails considering dependent lifetimes in the series system model mentioned earlier. We consider this

approach circuitous. A proper framework for discussing dependent competing risks requires that the $H_i(t)$'s be random; a comprehensive way of doing this is to assume a stochastic process model $\{H_i(t); t \geq 0\}$, $i = 1, \dots, k$, as was done in Section 6. We call such a model a competing-risk process, and call the k -variate process $\{H_1(t), \dots, H_k(t); t \geq 0\}$ a dependent competing-risk process if the $H_i(t)$'s are interdependent. A unit fails when any one of the k marginal processes $\{H_i(t); t \geq 0\}$, $i = 1, \dots, k$, hits the item's HP X . Interdependence of the $H_i(t)$'s will induce dependence between the corresponding lifetimes T_i , $i = 1, \dots, k$. Thus the prevailing notion of what constitutes dependent competing risks will be sustained, albeit more as a consequence than as a fundamental construct. Viewing the competing-risk scenario from the standpoint of hitting the HP offers a convenient platform for appreciating the phenomenon of lifetimes under dependent competing risks.

Having stated the foregoing, the question still remains as to what would be suitable models for the k -variate process $\{H_1(t), \dots, H_k(t); t \geq 0\}$, where the marginal processes $\{H_i(t); t \geq 0\}$, $i = 1, \dots, k$, are such that each $H_i(t)$ is nondecreasing in t . One possibility would be to let each marginal process be a Brownian maximum process of Section 6.2.1 and deduce the interdependence between the marginal processes from the assumed dependence of the k -variate Brownian motion process that generate Brownian maxima processes. The specifics remain to be worked out. Another possibility, in the case where $k = 2$, is to assume that $\{H_1(t); t \geq 0\}$ is a nonnegative, nondecreasing, and right-continuous process of the type discussed in Section 6.2, but that the sample path of $\{H_2(t); t \geq 0\}$ is an impulse function of the form $H_2(t) = 0$ for all $t \neq t^*$, and $H_2(t^*) = \infty$, for some $t = t^* > 0$, where the rate of impulse occurrence depends on the state of the process $\{H_1(t); t \geq 0\}$. Such a model may be meaningful when the process $\{H_1(t); t \geq 0\}$ can be identified with, say, degradation and the process $\{H_2(t); t \geq 0\}$ can be identified with some form of trauma with a rate of occurrence depending on the state of the degradation process. Here degradation and trauma compete with each other for the lifetime of the system. Lemoine and Wenocur (1985) and Wenocur (1989) have proposed the foregoing as a framework for failure modeling, although not in the context of competing risks. With appropriate modifications, their results could be adapted for the competing-risk scenario.

7.2 Degradation and Aging Processes

Much has been written on what is known as "degradation modeling" and reliability assessment using degradation data. The thinking here has been that degradation is an observable phenomenon and that failure occurs when the level of degradation hits some threshold (see Doksum 1991). What the threshold should be and how it should be specified has not been made clear. Our review of the engineering and materials science literature on degradation suggests that this viewpoint is questionable. This is because degradation is viewed as the irreversible accumulation of damage throughout life that ultimately leads to failure (see Bogdanoff and Kozin 1985, p. 1). Whereas the term "damage" itself is not defined, it is claimed that damage manifests as cracks, corrosion, physical wear (depletion of material), and so on. Similarly, with regard to aging, a review of the literature on longevity and mortality indicates that aging pertains

to a unit's position in a state space in which the probabilities of failure are greater than in a former position and that the manifestations of aging are the biomedical and physical difficulties experienced by older individuals.

Thus it appears that both degradation and aging are abstract constructs that cannot be observed and thus cannot be measured. However, these constructs serve to describe a process that results in failure and can be viewed as the cause of observables such as crack growth and corrosion, which can be measured. Thus the question arises as to how one can mathematically model the degradation phenomenon and relate it to the observables mentioned earlier. Put another way, how can we mathematically describe the cause and effect phenomenon of degradation and the observables that it spawns? Our proposal is to treat the former as a cumulative hazard process and the latter as a covariate (or a marker) process that is influenced by the former (see, e.g., Whitmore, Crowder, and Lawless 1998). This viewpoint of view may fit well with Aalen's (1987) proposal that matters of causality be handled by stochastic process models. As before, the item fails when the cumulative hazard process hits the item's HP X . With the foregoing in mind, we define a degradation process as a bivariate stochastic process $\{H(t), Z(t); t \geq 0\}$, with $H(t)$ representing the unobserved cumulative hazard, and $Z(t)$ representing an observable marker that is a precursor to failure. In principle, $\{Z(t); t \geq 0\}$, the marker process, can also be a vector stochastic process. Whereas $H(t)$ is required to be nondecreasing, there is no such restriction on $Z(t)$; cracks can be repaired and sometimes do heal.

7.2.1 Specifying Degradation Processes. When the marker process can be meaningfully described by a Markov process, for which there is some precedence when the marker is crack growth (see Sobczyk 1987), the degradation process $\{H(t), Z(t); t \geq 0\}$ can be taken to be Cinlar's (1972) Markov additive process (MAP). When this is the case, $\{H(t); t \geq 0\}$ is a Lévy process with parameters depending on the state of the $\{Z(t); t \geq 0\}$ process. Another way to link the two processes in question is to use Cox's (1972) proportional hazards model or Aalen's (1989) additive hazards model, in which linkage is achieved through the processes $\{h(t); t \geq 0\}$ and $\{Z(t); t \geq 0\}$. The ramifications of the foregoing, as well as the MAP, remain to be explored. Our main purpose here is to propose a different approach for examining the degradation phenomenon and the role of the HP in analyzing it.

8. THE HAZARD GRADIENT AND CONDITIONAL HAZARD POTENTIALS

The purposes of this section are to obtain a generalization of Theorem 1 and to further explore the ramifications of dependent life-lengths and dependent HPs. We start with the notion of a "hazard gradient" and provide a strategy through which a collection of dependent lifetimes can be replaced by a collection of independent ones.

Let T_1, \dots, T_n , be a collection of n lifetimes, and let $P(T_1 \geq t_1, \dots, T_n \geq t_n) = R(t_1, \dots, t_n)$ be its survival function. Let $\mathbf{t} = (t_1, \dots, t_n)$ be such that $R(\mathbf{t}) > 0$. The quantity $H(\mathbf{t}) = \ln R(\mathbf{t})$ is the multivariate analog of $H(t)$. Suppose that $H(\mathbf{t})$

has a gradient $\mathbf{r}(\mathbf{t}) = (r_1(\mathbf{t}), \dots, r_n(\mathbf{t}))$, where $r_i(\mathbf{t}) = \frac{\partial}{\partial t_i} H(\mathbf{t})$, $i = 1, \dots, n$. The quantity $\mathbf{r}(\mathbf{t})$ is called the hazard gradient of $R(\mathbf{t})$ (see Marshall 1975a).

The relationship among $H(\mathbf{t})$, $R(\mathbf{t})$, and $\mathbf{r}(\mathbf{u})$ is expressed through

$$H(\mathbf{t}) = \int_0^{\mathbf{t}} \mathbf{r}(\mathbf{u}) d\mathbf{u} \quad (26)$$

and

$$P(T_1 \geq t_1, \dots, T_n \geq t_n) = \exp\left(-\int_0^{\mathbf{t}} \mathbf{r}(\mathbf{u}) d\mathbf{u}\right). \quad (27)$$

Marshall (1975a) gave a decomposition of $H(\mathbf{t})$ that is noteworthy due to its role in allowing us to prove Theorem 5. Specifically,

$$H(\mathbf{t}) = \int_0^{t_1} r_1(u_1, 0, \dots, 0) du_1 + \int_0^{t_2} r_2(t_1, u_2, 0, \dots, 0) du_2 + \dots + \int_0^{t_n} r_n(t_1, \dots, t_{n-1}, u_n) du_n, \quad (28)$$

where $r_1(u_1, 0, \dots, 0)$ is the failure rate of T_1 at u_1 , and $r_i(t_1, \dots, t_{i-1}, u_i, 0, \dots, 0)$ is the (conditional) failure rate of T_i at u_i , were it so that $T_1 > t_1, \dots, T_{i-1} > t_{i-1}$.

The first term on the right side of (28) is the cumulative hazard of T_1 at t_1 and is denoted by $H_1(t_1)$. The second term is the integral of the conditional hazard of T_2 at u_2 given that $T_1 \geq t_1$; it is denoted by $H_2(t_2|t_1)$. Similarly, the last term is denoted by $H_n(t_n|t_1, \dots, t_{n-1})$. Thus

$$H(\mathbf{t}) = H_1(t_1) + H_2(t_2|t_1) + \dots + H_n(t_n|t_1, \dots, t_{n-1}),$$

and because $R(\mathbf{t}) = \exp(-H(\mathbf{t}))$,

$$P(T_1 \geq t_1, \dots, T_n \geq t_n) = \exp[-H_1(t_1)] \exp[-H_2(t_2|t_1)] \dots \times \exp[-H_n(t_n|t_1, \dots, t_{n-1})]. \quad (29)$$

Clearly, $e^{-H_1(t_1)} = P(T_1 \geq t_1)$, and, using arguments that parallel those leading us to (1), we can see that for any $n \geq 2$,

$$\exp[-H_n(t_n|t_1, \dots, t_{n-1})] = P(T_n \geq t_n | T_1 \geq t_1, \dots, T_{n-1} \geq t_{n-1}). \quad (30)$$

Let X_1, \dots, X_n , be the HPs corresponding to the lifetimes T_1, \dots, T_n and the cumulative hazards $H_1(t_1), \dots, H_n(t_n)$. Then, a consequence of the relationship (29) is that

$$\begin{aligned} P(T_n \geq t_n | T_1 \geq t_1, \dots, T_{n-1} \geq t_{n-1}) \\ = P(X_n \geq H_n(t_n) | X_1 \geq H_1(t_1), \dots, X_{n-1} \geq H_{n-1}(t_{n-1})) \\ = \exp[-H_n(t_n|t_1, \dots, t_{n-1})]. \end{aligned} \quad (31)$$

Because T_1, \dots, T_n are not independent, the HPs X_1, \dots, X_n are, by virtue of Remark 1, also not independent. However, the hand side of (31) is the distribution function of an exponentially distributed random variable, say X_n^* , with a scale parameter of 1, evaluated at $H_n(t_n|t_1, \dots, t_{n-1})$. Thus, from (30), we have the result that for all $n \geq 2$,

$$\begin{aligned} P(T_n \geq t_n | T_1 \geq t_1, \dots, T_{n-1} \geq t_{n-1}) \\ = P(X_n \geq H_n(t_n) | X_1 \geq H_1(t_1), \dots, X_{n-1} \geq H_{n-1}(t_{n-1})) \\ = P(X_n^* \geq H_n(t_n|t_1, \dots, t_{n-1})). \end{aligned}$$

The quantity X_n^* is called the conditional HP of the n th item; its unit exponential distribution is indexed by $H_n(t_n|t_1, \dots, t_{n-1})$. In contrast, X_n , the HP of the n th item, has a unit exponential distribution indexed by $H_n(t_n)$.

Similarly, corresponding to each term on the right side of (28) except the first, there exist random variables X_2^*, \dots, X_{n-1}^* , independent of one another, and also of X_n^* , such that

$$\begin{aligned} P(T_1 \geq t_1, \dots, T_n \geq t_n) \\ = P(X_1 \geq H_1(t_1))P(X_2^* \geq H_2(t_2|t_1)) \cdots \\ \times P(X_n^* \geq H_n(t_n|t_1, \dots, t_{n-1})). \end{aligned}$$

We have now proved, as a multivariate analog to Theorem 1, the following results.

Theorem 5. Corresponding to every collection of nonnegative variables T_1, \dots, T_n , having a survival function $R(t_1, \dots, t_n)$, there exists a collection of n independent and exponentially distributed random variables X_1, X_2^*, \dots, X_n^* , with scale parameter 1; X_1 is indexed on $H_1(t_1)$, and for $n \geq 2$, X_n^* is indexed on $H_n(t_n|t_1, \dots, t_{n-1})$.

9. SUMMARY

In this article we have described a unifying perspective on the process leading to the failure of items that is context-independent. This perspective is made possible through the notion of an HP. Besides providing an alternative means of conceptualizing the failure process, the HP provides a means by which the nature of dependence between the lifetimes can be understood and exploited. With respect to the latter, we can generate (new) families of multivariate failure distributions using multivariate exponentials with unit exponential marginals as seeds. For items required to operate in dynamic environments, the HP provides a vehicle by which new families of univariate survival functions can be obtained. This is achieved by establishing a connection between the failure process and the killing times of continuous and increasing stochastic processes to a random barrier, which is the HP. The notion of a HP generalizes to a nonexponential distribution for the barrier and also to the multivariate case. To conclude, the importance of the notion of a HP stems from its ability to provide a different perspective on failure, a model for the cause of dependence of lifetimes, new multivariate models for failure, new univariate models for survival in dynamic environments, and a perspective on competing risks and degradation modeling.

This article is expository in the sense that it provides a feel for the foregoing possibilities. Clearly, more can be done. For one, stochastic processes other than those considered in Section 6.2 can be investigated. We may do more on considering covariates that drive the $\{H(t); t \geq 0\}$ process. Another possibility would be to consider bivariate processes and their killing times by interdependent barriers. In regard to the latter, one may also be able to leverage the idea for assessing competing risks by looking at the bivariate cumulative hazard process. Finally, there is a matter of statistical inference and model validation, topics that have not been touched on here. The possibilities of further capitalizing the notion of an HP are promising for reliability theorists, survival analysts, and actuarial scientists.

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REFERENCES

- Aalen, O. O. (1987), "Dynamic Modeling and Causality," *Scandinavian Actuarial Journal*, 12, 177–190.
- (1989), "A Linear Regression Model for the Analysis of Life Times," *Statistics in Medicine*, 8, 907–925.
- Aven, T., and Jensen, U. (1999), *Stochastic Models in Reliability*, New York: Springer-Verlag.
- Bagdonavicius, V., and Nikulin, M. S. (1999), "Model Building in Accelerated Experiments," in *Statistical and Probabilistic Models in Reliability*, eds. D. C. Ionescu and N. Limnios, Boston: Birkhauser, pp. 51–73.
- Barlow, R. E., and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing*, New York: Holt, Rinehart and Winston.
- Blumenthal, R. M., Getoor, R. K., and McKean, H. P. (1962), "Markov Processes With Identical Hitting Distributions," *Illinois Journal of Mathematics*, 6, 402–420.
- Bogdanoff, J. L., and Kozin, F. (1985), *Probabilistic Models of Cumulative Damage*, New York: Wiley.
- Cinlar, E. (1972), "Markov Additive Processes II," *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 24, 94–121.
- (1979), "On Increasing Continuous Processes," *Stochastic Processes and Their Applications*, 9, 147–154.
- Cinlar, E., and Ozekici, S. (1987), "Reliability of Complex Devices in Random Environments," *Probability in the Engineering and Informational Sciences*, 1, 97–115.
- Cox, D. R. (1972), "Regression Models and Life Tables" (with discussion), *Journal of the Royal Statistical Society, Ser. B*, 34, 187–220.
- Doksum, K. A. (1991), "Degradation Models for Failure Time and Survival Data," *CWI Quarterly, Amsterdam*, 4, 195–203.
- Duchesne, T., and Rosenthal, J. S. (2003), "On the Collapsibility of Lifetime Regression Models," *Advances in Applied Probability*, 35, 755–772.
- Durham, S. D., and Padgett, W. J. (1997), "A Cumulative Damage Model for System Failure With Application to Carbon Fibers and Composites," *Technometrics*, 39, 34–44.
- Esary, J. D., Marshall, A. W., and Proschan, F. (1973), "Shock Models and Wear Processes," *The Annals of Applied Probability*, 1, 627–649.
- Gavrilov, L. A., and Gavrilova, N. S. (2001), "The Reliability Theory of Aging and Longevity," *Journal of Theoretical Biology*, 213, 527–545.
- Gjessing, H. K., Aalen, O. O., and Hjort, N. (2003), "Frailty Models Based on Lévy Processes," *Advances in Applied Probability*, 35, 532–550.
- Grandell, J. (1975), *Doubly Stochastic Poisson Processes*, New York: Springer-Verlag.
- Gumbel, E. J. (1960), "Bivariate Exponential Distributions," *Journal of the American Statistical Association*, 55, 698–707.
- Kebir, Y. (1991), "On Hazard Rate Processes," *Naval Research Logistics*, 38, 865–876.
- Kingman, J. F. C. (1964), "On Doubly Stochastic Poisson Processes," *Proceedings of the Cambridge Philosophical Society*, 60, 923–960.
- Kotz, S., Balakrishnan, N., and Johnson, N. L. (2000), *Continuous Multivariate Distributions*, New York: Wiley.
- Lemoine, A. J., and Wenocur, M. L. (1985), "On Failure Modeling," *Naval Research Logistics Quarterly*, 32, 497–508.
- Lindley, D. V., and Singpurwalla, N. D. (1986), "Multivariate Distributions for the Lifetimes of Environment," *Journal of Applied Probability*, 23, 418–431.
- Marshall, A. W. (1975a), "Some Comments on the Hazard Gradient," *Stochastic Processes and Their Applications*, 3, 293–300.
- (1975b), "Multivariate Distributions With Monotone Hazard Rate," in *Reliability and Fault Tree Analysis*, eds. J. B. Fussell and N. D. Singpurwalla, Philadelphia: Society for Industrial and Applied Mathematics, pp. 259–284.
- Marshall, A. W., and Olkin, I. (1967), "A Multivariate Exponential Distribution," *Journal of the American Statistical Association*, 62, 30–44.
- Nelson, R. B. (1995), "Copulas, Characterization, Correlation and Counterexamples," *Mathematics Magazine*, 68, 193–198.
- Pettit, L. I., and Young, K. D. S. (1999), "Bayesian Analysis for Inverse Gaussian Lifetime Data With Measures of Degradation," *Journal of Statistical Computation and Simulation*, 63, 217–234.
- Pitman, J. W., and Speed, T. P. (1973), "A Note on Random Times," *Stochastic Processes and Their Applications*, 1, 369–374.
- Protter, P. (1990), *Stochastic Integration and Differential Equations: A New Approach*, Berlin: Springer-Verlag.
- Sedyakin, N. M. (1966), "On One Physical Principle in Reliability Theory," *Technical Cybernetics*, 3, 80–87.
- Singpurwalla, N. D. (1995), "Survival in Dynamic Environments," *Statistical Science*, 10, 86–103.
- Singpurwalla, N. D., and Kong, C. W. (2004), "Specifying Interdependence in Networked Systems," *IEEE Transactions in Reliability*, 53, 401–405.

- Singpurwalla, N. D., and Wilson, S. P. (1995), "The Exponentiation Formula of Reliability and Survival: Does It Always Hold?" *Lifetime Data Analysis*, 1, 187–194.
- Singpurwalla, N. D., and Youngren, M. A. (1993), "Multivariate Distributions Induced by Dynamic Environments," *Scandinavian Journal of Statistics*, 20, 251–261.
- Sobczyk, K. (1987), "Stochastic Models for Fatigue Damage of Materials," *Advances in Applied Probability*, 19, 652–673.
- Suppes, P. (1970), *A Probabilistic Theory of Causality*, Amsterdam: North-Holland.
- Wenocur, M. L. (1989), "A Reliability Model Based on Gamma Process and Its Analytic Theory," *Advanced Applied Probability*, 21, 899–918.
- Whitmore, G. A., Crowder, M. J., and Lawless, J. F. (1998), "Failure Inference From a Marker Process Based on a Bivariate Wiener Process," *Lifetime Data Analysis*, 4, 229–251.
- Yang, Y., and Klutke, G. A. (2000), "Lifetime Characteristics and Inspection Schemes for Lévy Degradation Processes," *IEEE Transactions on Reliability*, 49, 377–382.
- Yashin, A., and Arjas, E. (1988), "A Note on Random Intensities and Conditional Survival Functions," *Journal of Applied Probability*, 25, 630–635.
- Yor, M. (1992), "On Some Exponential Functionals of Brownian Motion," *Advances in Applied Probability*, 24, 509–531.