Signal Processing

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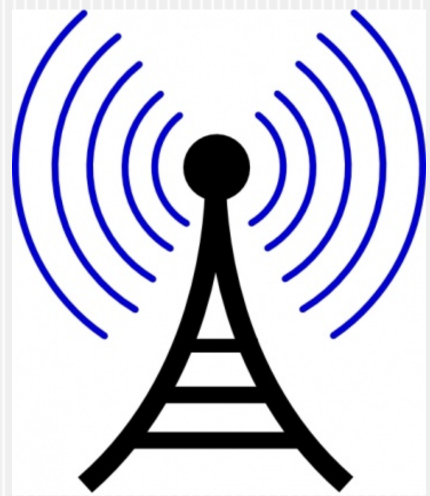
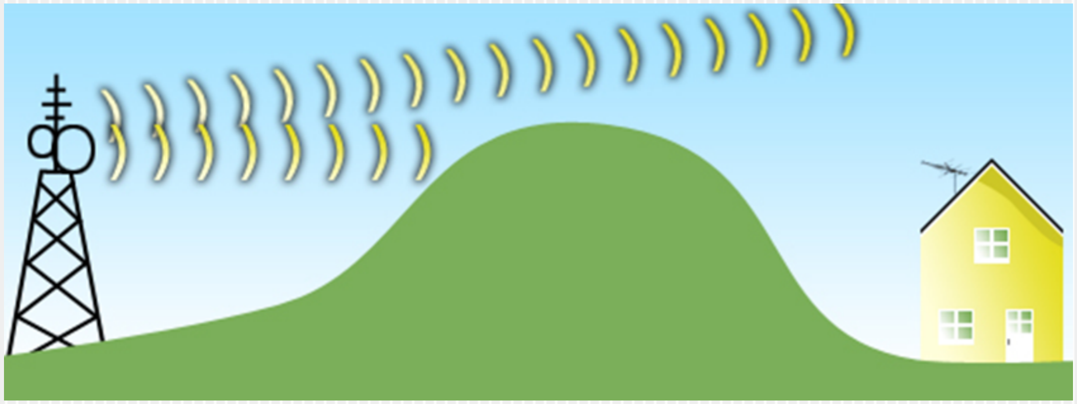
What is a signal?

- A **signal** is a phenomenon evolving over time or space.
- Any time-varying or spatial-varying quantity, i.e. any quantity measurable through time or over space.
- Examples:
  - Sounds
  - Images
  - Text messages



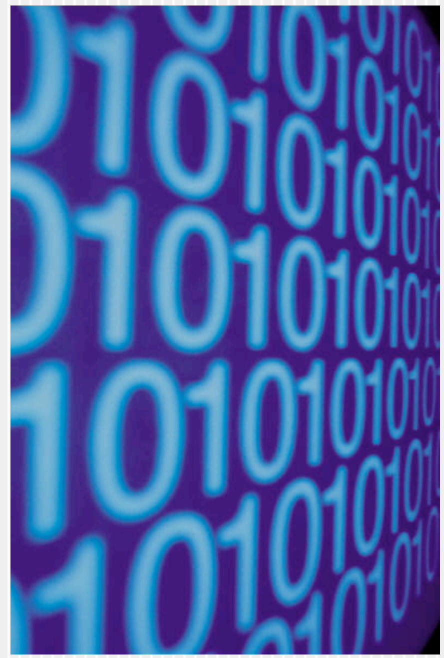
Signal Processing

- **Signal processing** is any manual or mechanical operation which modifies, analyzes or otherwise manipulates the information contained in a signal.
  - Signal processing operates on an abstract representation of a physical quantity and not on the quantity itself.
- **Signal transmission** is the process of sending and receiving information through signals.



Types of Signal Processing

- In **Digital Signal Processing,** everything is described in terms of integers, regardless of the signal's origin.



Primitive Digital Signal Processing

- Palermo Stone:
  - Egypt, 2500 BC
  - Nile flooding patterns recorded through use of “nilometers”
    - Very accurate
    - Considered first recorded digital signal that is still relevant today
- Limited in applicability

Analog Signal Processing

- The **Analog** (*continuum*) world model is the idea that time and space are an uninterrupted flow which can be divided arbitrarily many times into arbitrarily and infinitely many pieces.

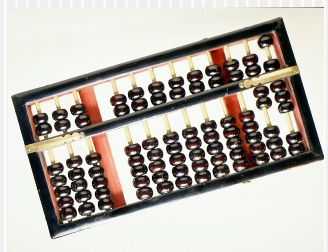
- Pythagoras discovers existence of irrational numbers
  - Idea of the *continuum*

- The analog signals can be understood as continuous functions

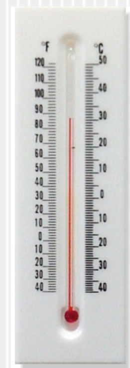
- Analog signals are often natural or physical phenomena such as:
  - Sound waves
  - Light waves
  - Temperature
  - Current or Voltage

Quiz: Analog vs Digital

Digital



Analog



Quiz: Analog vs Digital

A telephone uses BOTH analog and digital signals!

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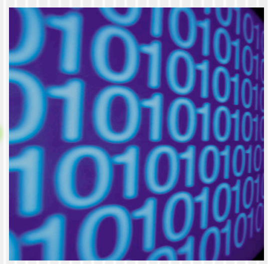
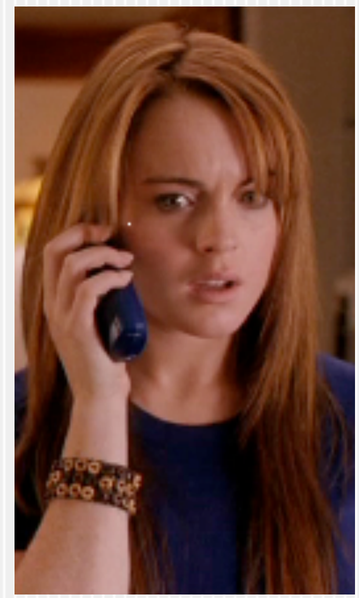


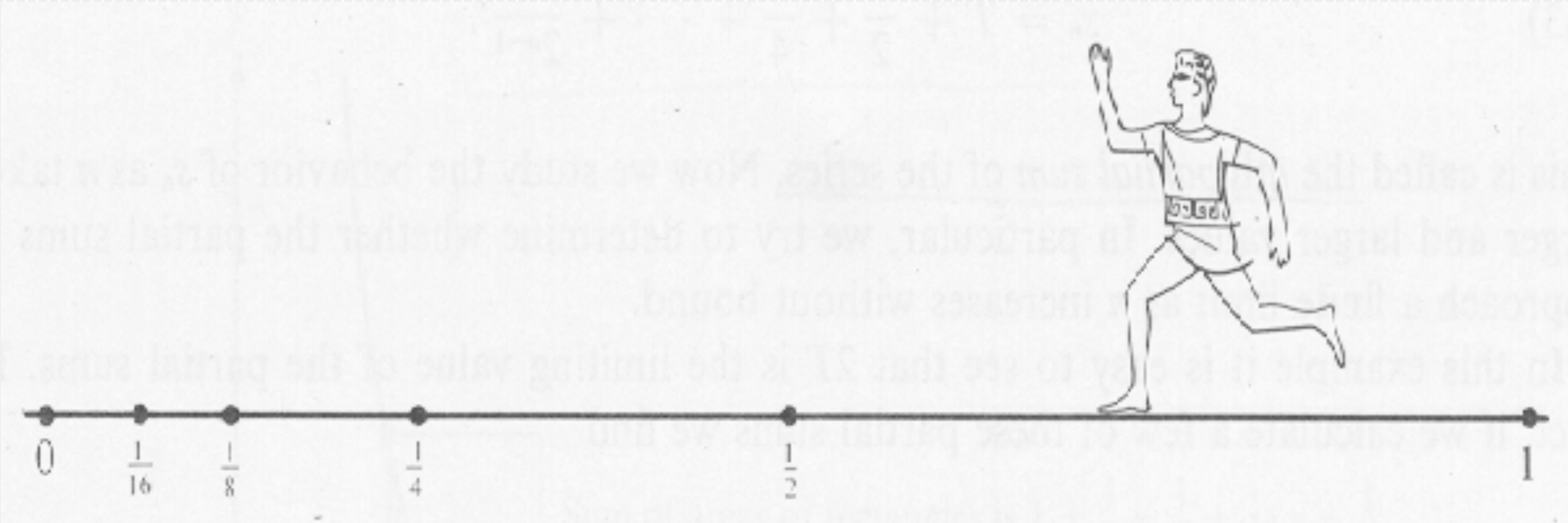
The analog signal (voice) is turned into digital signal



The digital signal is then reconstructed into sound waves.

Zeno's Paradox

- The dichotomy paradox – can you ever get out the door?
- Highlights the gap between the digital and analog views of the world.

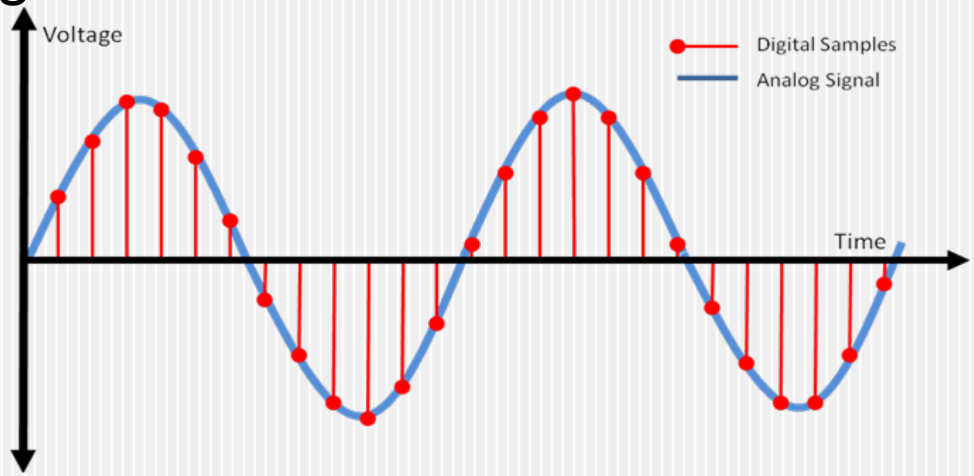



Zeno of Elea,
How come you never take me out anymore?
Why bother... movement is a logical impossibility.
S.O.

Discrete Time Sampling

- We gather data through **sampling** - measuring the quantity of interest at regular intervals.

- Sampling is done by:
  - Measuring machines
  - Humans



- But these can never take infinite amount of samples in a finite interval (remember Zeno?)

- Analog phenomenon is continuous whereas digital is discrete… So how do we bridge this gap?

Continuous vs. Discrete

Continuous	Discrete
• Consider computing the average temperature over a certain time interval. • Using Calculus, we can calculate **exact** average $\bar{C}$ given temperature function $f(t)$ over interval $[T_0, T_1]$: $$\bar{C} = \frac{1}{T_1 - T_0} \int_{T_0}^{T_1} f(t)dt$$	• Now, consider the case in which all we have is a set of daily measurements c_1, $c_2, \ldots, c_D$. • We compute average temperature of our measurements over D: $$\hat{C} = \frac{1}{D} \sum_{n=1}^{D} c_n$$

So, how different is $\hat{C}$ different from $\bar{C}$? Clearly $\bar{C}$ is abstractly more accurate, however we can only measure discretely in real life.

Bridging Gap between Discrete and Continuous

- First, assume that a temperature function $f(t)$ exists.
- Then the measured values c_n are samples of the function taken one day apart:

$$c_n = f(nT_s)$$

where T_s is the duration of a day.

- Then, the sum is just the Riemann approximation to the integral.

Narrowed Question : So, how good is our approximation? How much information are we discarding by only keeping samples of a continuous-time function? Are they equivalent?

Equivalence representation

- If our physical phenomenon *“doesn’t change too fast,”* then we can conclude that the continuous-time function and the set of samples are perfectly equivalent representations.

- This equivalence only holds for signals which are sufficiently slow with respect to how fast we sample them. This is good because we prefer the signals to be slow to avoid any “craziness” between successive samples.

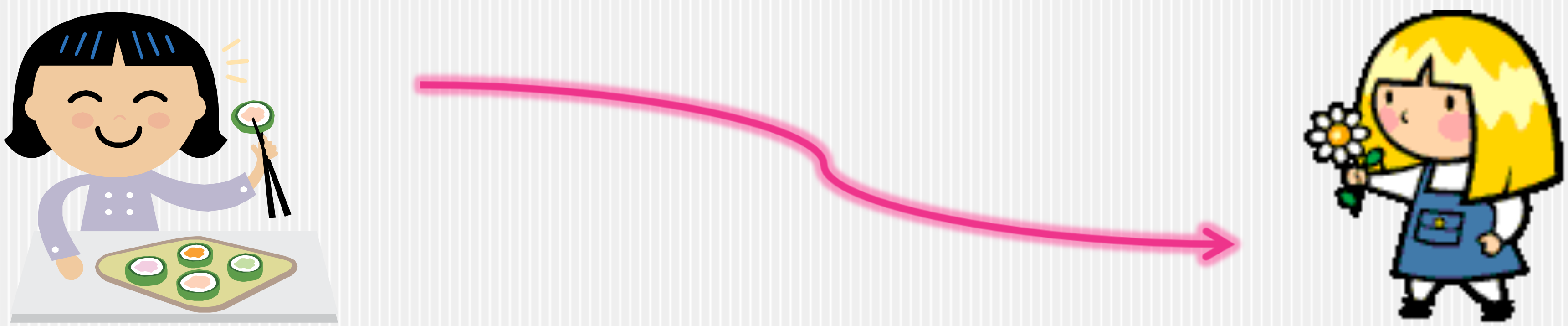
- Thus, the analog and the digital world can perfectly coexist!

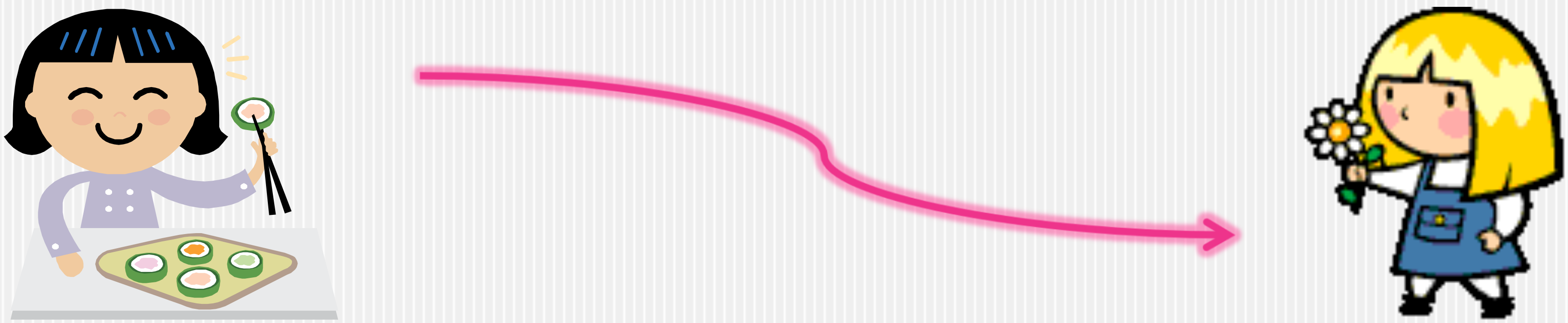
Signal Transmission using Power Series

We know that all types of signal transmission are based on sending a series of ***numbers.***

Suppose we know that a signal is given as a function *f* with a power series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$

- the set of coefficients $\{a_n\}_{n=0}^{\infty}$ is the important information that needs to be **sent** and so the receiver can **reconstruct** the signal





Hyesu wants to send Kim the graph of f, where $f(x) = \sum_{n=0}^{\infty} a_n x^n$:

1. Hyesu finds coefficients a_0, a_1, a_2 ...
2. Hyesu sends a_0, a_1, a_2 ... and Kim receives them
3. Kim reconstructs the signal by multiplying the coefficients a_n by x^n and forming the original infinite series $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Is it really that simple??

Restrictions

- Problem #1: Hyesu cannot determine nor send an *infinite* number of coefficients
  - She can send the *finite* sequence $a_0, a_1, \dots, a_N$
  - Thus she can transmit the signal $\sum_{n=0}^{N} a_n x^n$ rather than $\sum_{n=0}^{\infty} a_n x^n$

- Problem #2: Hyesu can only send the coefficients with a certain level of *precision*
  - She can send some numbers $\widetilde{a_0} \approx a_0, \widetilde{a_1} \approx a_1, \dots, \widetilde{a_N} \approx a_N$
  - She ultimately transmits the signal $\tilde{f}(\mathrm{x}) = \sum_{n=0}^{N} \widetilde{a_n} x^n$ to Kim.

What can Hyesu do to make sure $\tilde{f}(\mathrm{x})$ is as close to $f(x)$ as possible?

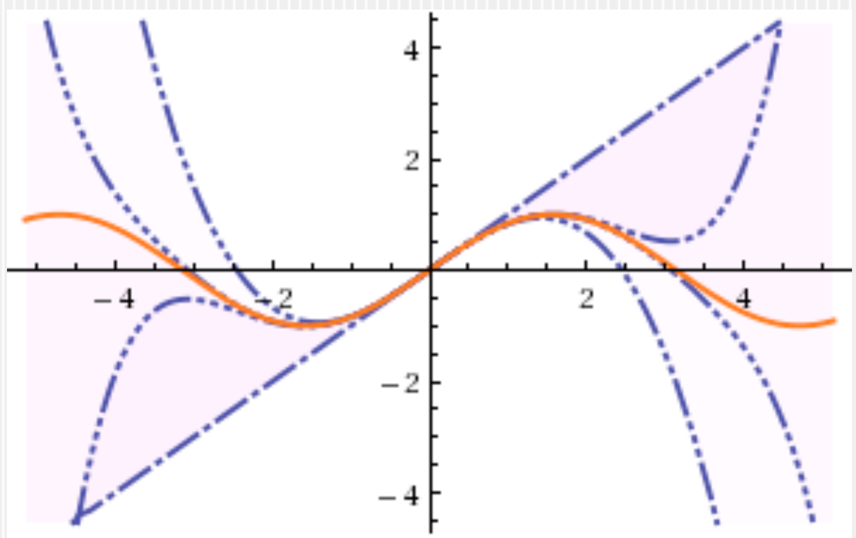
Making $\tilde{f}(x)$ the best approximation of $f(x)$:

- Strategy : Send the most important (i.e. largest) coefficients
  - Sending the *largest* N+1 coefficients (rather than just the *first* N+1) helps determine the "best N-term approximation"

Example: $f(x) = \sin(x) = 0 + \mathrm{x} + 0\mathrm{x}^2 - \frac{1}{6}\mathrm{x}^3 + 0\mathrm{x}^4 + \frac{1}{120}\mathrm{x}^5 + \ \ldots$

First 4 coefficients: {0, 1, 0, -.1666}; **Largest** 4 coefficients: {1, -.1666, .0083, -0.000198}

The reconstructed signal is closer to the original when we send the **larger** coefficients.



(order n approximation shown with n dots)

Which is better…

Frames

Or

Bases?

Frames in Quantization

- **Quantization** is the process of mapping a large set of input values to a smaller set , in doing so there will be loss of precision.

- Suppose we are sending a signal using the Mercedes-Benz frame, and suppose we perturb our frame coefficients by adding in **white noise**, w_i with expected value of 0 and variance of σ^2

- With this noise factoring into the accuracy of our signal transmission, we want to consider the error introduced to the reconstruction due to the noise.

- We can do this using this equation:

$$x - \hat{x} = \frac{2}{3}\sum_{i=1}^{3}\langle \varphi_i, x\rangle \varphi_i - \frac{2}{3}\sum_{i=1}^{3}(\langle \varphi_i, x\rangle + w_i)\varphi_i$$

Frames in Quantization

Error in reconstruction of the signal with the introduction of noise

$$x - \hat{x} = \underbrace{\frac{2}{3}\sum_{i=1}^{3}\langle \varphi_i, x\rangle \varphi_i}_{\text{Reconstruction of } x \text{ using the Mercedes-Benz frame}} - \overbrace{\frac{2}{3}\sum_{i=1}^{3}(\langle \varphi_i, x\rangle + w_i)\varphi_i}^{\text{Reconstruction of } x \text{ when considering the noise factor}}$$

Noise factor

Frames in Quantization

- Simplified version of error:

$$x - \hat{x} = -\frac{2}{3}\sum_{i=1}^{3} w_i \varphi_i$$

- Using this equation, we can find the average error per component to compare to the error we get from using an orthonormal basis to transmit our signal.

$$\mathrm{MSE} = \frac{1}{2} E\left\| x - \hat{x} \right\| = \frac{1}{2} E\left\| \frac{2}{3}\sum_{i=1}^{3} w_i \varphi_i \right\|^2$$

$$= \frac{1}{2}\sigma^2 \frac{4}{9}\sum_{i=1}^{3} \left\| \varphi_i \right\|^2 = \frac{2}{3}\sigma^2$$

Frames Win!

Error obtained with:	
Orthonormal basis	Mercedes-Benz Frame
σ^2	$(2/3)\sigma^2$

- Clearly we can see that by using a frame as opposed to an orthonormal basis, there is less possibility of error in reconstruction.

Sources

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