

HYDRODYNAMIC FORCES

Drag Force and Drag Coefficient

A particle suspended in a fluid is subjected to hydrodynamic forces. For low Reynolds' number, the Stokes drag force on a spherical particle is given by

$$F_D = 3\pi\mu Ud, \quad (1)$$

where d the particle diameter, μ is the coefficient of viscosity and U is the relative velocity of the fluid with respect to the particle. Equation (1) may be restated as

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A} = \frac{24}{Re} \quad (2)$$

In Equation (3), ρ is the fluid (air) density, $A = \frac{\pi d^2}{4}$ is cross sectional area of the spherical particle, and

$$Re = \frac{\rho U d}{\mu} \quad (3)$$

is the Reynolds number. The Stokes drag is applicable to the creeping flow regime (Stokes regime) with small Reynolds numbers ($Re < 0.5$). At higher Reynolds numbers, the flow the drag coefficient deviates from Equation 2. Figure 1 shows the variation of drag coefficient for a sphere for a range of Reynolds numbers.

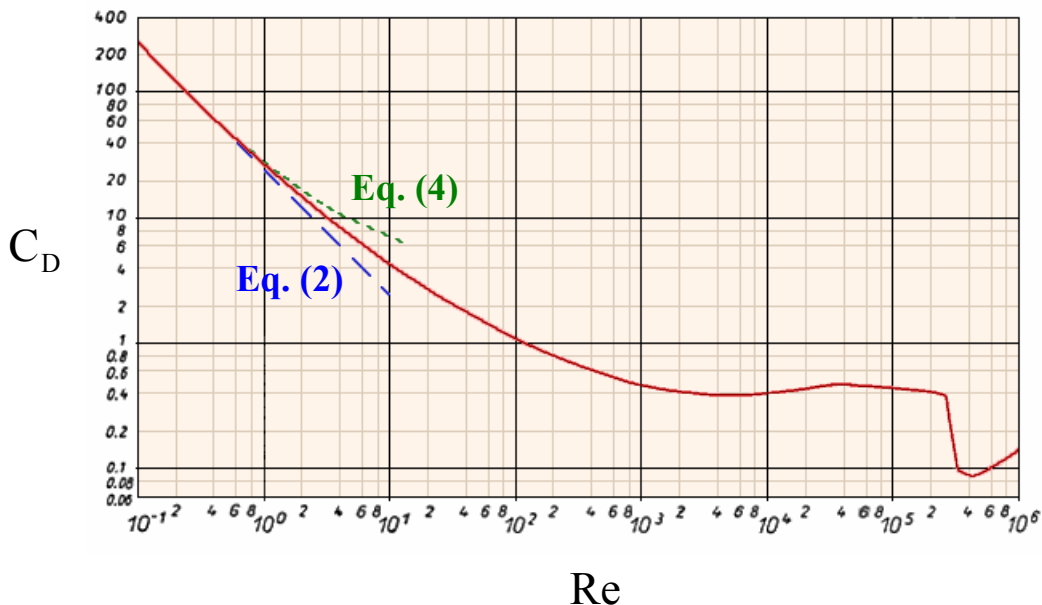


Figure 1. Variations of drag coefficient with Reynolds number for a spherical particle.

Oseen included the inertial effect approximately and developed a correction to the Stokes drag given as

$$C_D = \frac{24[1 + 3 Re/16]}{Re}, \quad (4)$$

which is shown in Figure 1.

For $1 < Re < 1000$, which is referred to as the transition regime, the following expressions may be used (Clift et al., 1978):

$$C_D = \frac{24[1 + 0.15 Re^{0.687}]}{Re}, \quad (5)$$

or

$$C_D = \frac{24}{Re} + \frac{4}{Re^{0.33}} \quad (6)$$

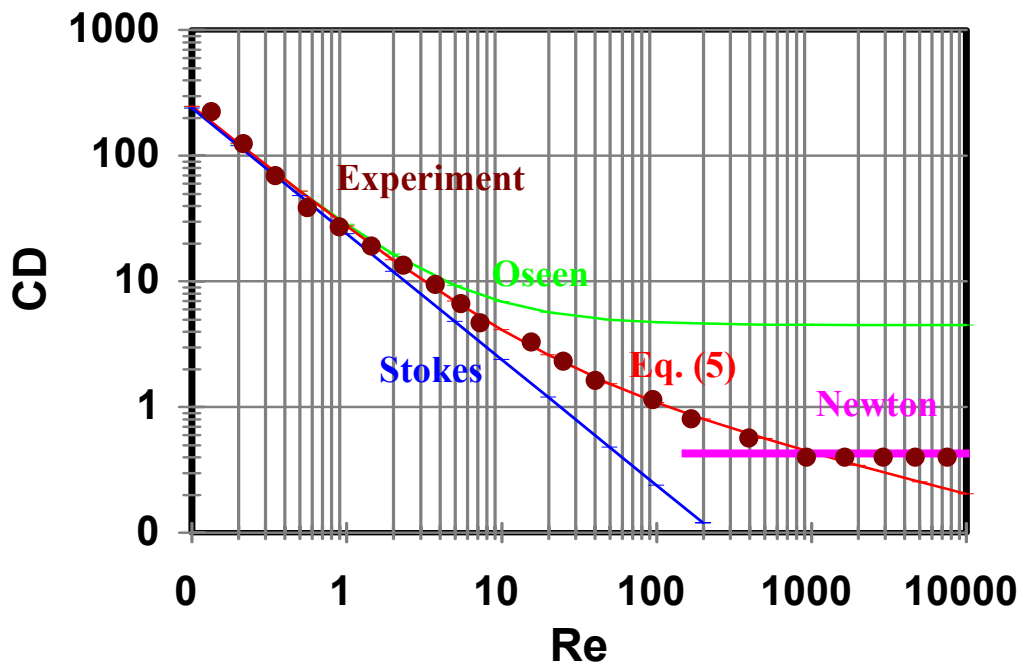


Figure 2. Predictions of various models for drag coefficient for a spherical particle.

For $10^3 < Re < 2.5 \times 10^5$, the drag coefficient is roughly constant ($C_D = 0.4$). This regime is referred to as the Newton regime. At $Re \approx 2.5 \times 10^5$, the drag coefficient decreases sharply due to the transition from laminar to turbulent boundary layer around the sphere. That causes the separation point to shift downstream as shown in Figure 3.

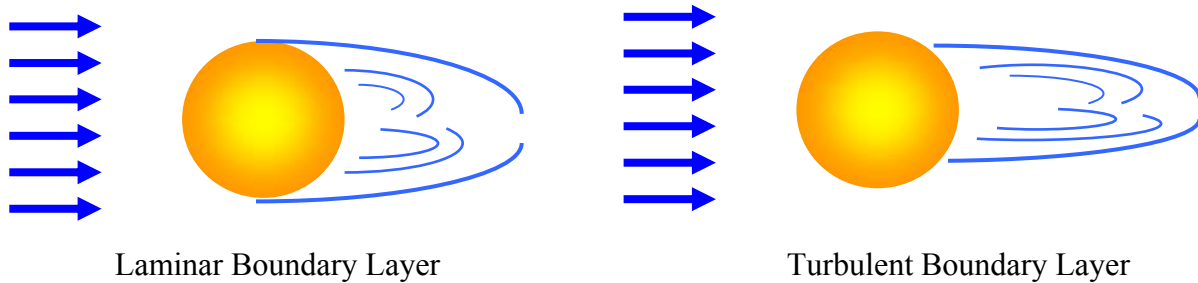


Figure 3. Laminar and turbulent boundary layer separation.

Wall Effects on Drag Coefficient

For a particle moving near a wall, the drag force varies with distance of the particle from the surface. Brenner (1961) analyzed the drag acting on a particle moving toward a wall under the creeping flow condition as shown in Figure 4a. To the first order, the drag coefficient is given as

$$C_D = \frac{24}{Re} \left(1 + \frac{d}{2h}\right) \quad (7)$$

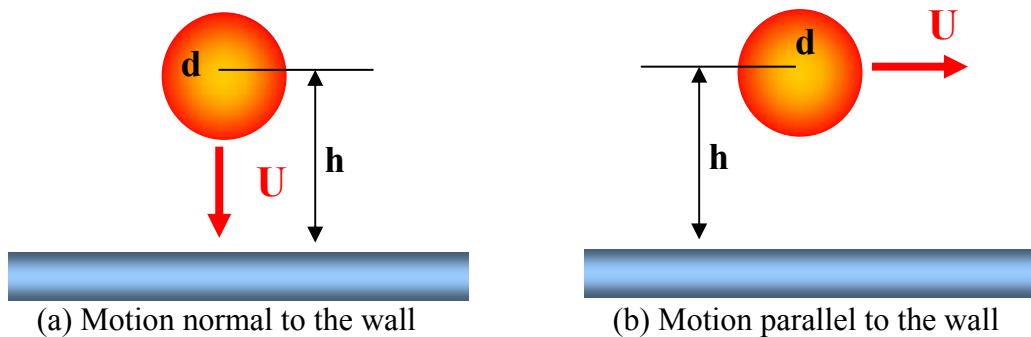


Figure 4. Particle motions near a wall.

For a particle moving parallel to the wall as shown in Figure 4b, the Stokes drag force need to be modifies. For large distances from the wall, Faxon (1923) found

$$C_D = \frac{24}{Re} \left[1 - \frac{9}{16} \left(\frac{d}{2h}\right) + \frac{1}{8} \left(\frac{d}{2h}\right)^3 - \frac{45}{256} \left(\frac{d}{2h}\right)^4 - \frac{1}{16} \left(\frac{d}{2h}\right)^5\right]^{-1} \quad (8)$$

Cunningham Correction Factor

For very small particles, when the particle size becomes comparable with the gas mean free path, slip occurs and the expression for drag must be modified accordingly. Cunningham obtained the needed correction to the Stokes drag force:

$$F_D = \frac{3\pi\mu U d}{C_c}, \quad (9)$$

where the Cunningham correction factor C_c is given by

$$C_c = 1 + \frac{2\lambda}{d} [1.257 + 0.4e^{-1.1d/2\lambda}] \quad (10)$$

Here λ denotes the molecular mean free path in the gas. Note that $C_c \geq 1$ for all values of d and λ . Figure 5 shows the variation of Cunningham correction factor with Knudsen number. It is seen that C_c is about 1 for $Kn < 0.1$ and increases sharply as Kn increases beyond 0.5. Table 4 illustrates the variation of Cunningham correction factor with particle diameter in air under normal pressure and temperature conditions with $\lambda = 0.07 \mu\text{m}$. Equation (10) is applicable to a wide range of $Kn = \frac{\lambda}{d} \leq 1000$ that covers slip, transition and part of free molecular flows. The particle Reynolds number and Mach number (bases on relative velocity), however should be small.

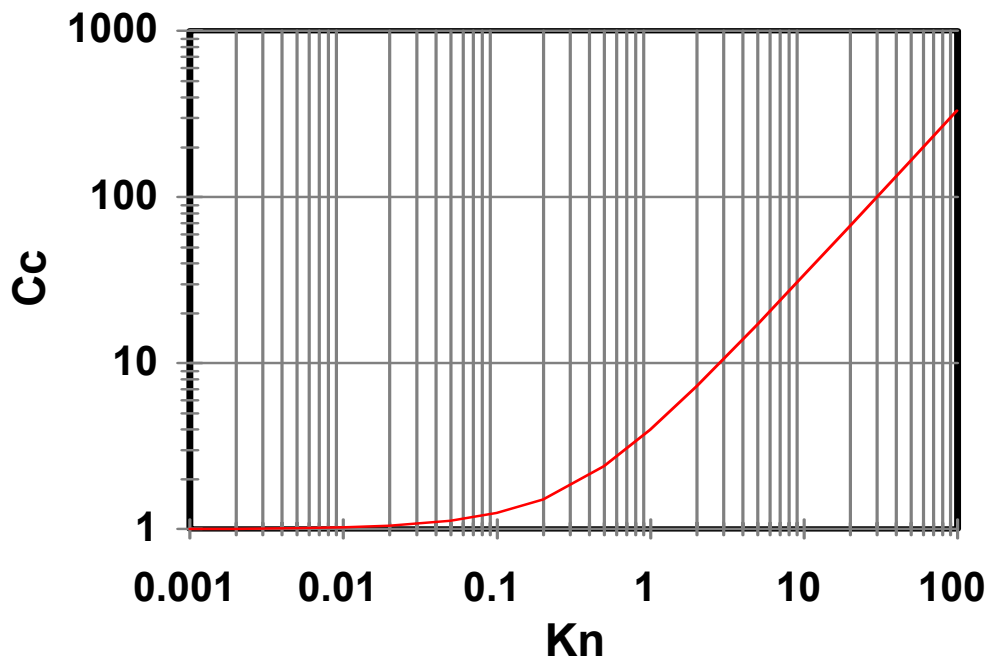


Figure 5. Variation of Cunningham correction with Knudsen number.

Table 4 – Variations of C_c with d for $\lambda = 0.07 \mu\text{m}$

Diameter, μm	C_c
10 μm	1.018
1 μm	1.176
0.1 μm	3.015
0.01 μm	23.775
0.001 μm	232.54

Compressibility Effect

For high-speed flows with high Mach number, the compressibility could affect the drag coefficient. Many expressions were suggested in the literature to account for the effect of gas Mach number on the drag force. Henderson (1976) suggested two expressions for drag force acting on spherical particles for subsonic and supersonic flows.

Accordingly, for subsonic flow

$$C_D = 24 \left[\text{Re} + S \left\{ 4.33 + 1.567 \times \exp\left(-0.247 \frac{\text{Re}}{S}\right) \right\} \right]^{-1} + \exp\left(-\frac{0.5M}{\sqrt{\text{Re}}}\right) \left[\frac{4.5 + 0.38(0.03\text{Re} + 0.48\sqrt{\text{Re}})}{1 + 0.03\text{Re} + 0.48\sqrt{\text{Re}}} + 0.1M^2 + 0.2M^8 \right] + \left[1 - \exp\left(-\frac{M}{\text{Re}}\right) \right] 0.6S \quad (11)$$

where M is Mach number based on relative velocity, $\Delta V = |V - V_p|$, and $S = M\sqrt{\frac{\gamma}{2}}$ is the molecular speed ratio, where γ is the specific heat ratio. For the supersonic flows with Mach numbers equal to or exceeding 1.75, the drag force is given by

$$C_D = \frac{0.9 + \frac{0.34}{M^2} + 1.86 \left(\frac{M}{\text{Re}}\right)^{\frac{1}{2}} \left[2 + \frac{2}{S^2} + \frac{1.058}{S} - \frac{1}{S^4} \right]}{1 + 1.86 \left(\frac{M}{\text{Re}}\right)^{\frac{1}{2}}} \quad (12)$$

For the flow regimes with Mach between 1 and 1.75, a linear interpolation is to be used.

Carlson and Hoglund (1964) proposed the following expression:

$$C_D = \frac{24}{Re} \frac{1 + \exp\left(-\frac{0.427}{M^{4.63}} - \frac{3}{Re^{0.88}}\right)}{1 + \frac{M}{Re} \left\{3.82 + 1.28 \exp\left(-1.25 \frac{Re}{M}\right)\right\}} \quad (13)$$

Droplets

For drag force for liquid droplets at small Reynolds numbers is given as

$$F_D = 3\pi\mu^f Ud \frac{1 + 2\mu^f / 3\mu^p}{1 + \mu^f / \mu^p} \quad (14)$$

where the superscripts f and p refer to the continuous fluid and discrete particles (droplets, bubbles), respectively.

Non-spherical Particles

For non-spherical (chains or fibers) particles, Stokes' drag law must be modified. i.e.,

$$F_D = 3\pi\mu Ud_e K, \quad (15)$$

where d_e is the diameter of a sphere having the same volume as the chain or fiber. That is,

$$d_e = \left(\frac{6}{\pi} \text{Volume}\right)^{1/3} \quad (16)$$

and K is a correction factor.

For a cluster of n spheres, $d_e = n^{1/3} d$. For tightly packed clusters, $k \leq 1.25$. Some other values of K are listed in Table 5.

Table 5 – Correction Coefficient

Cluster Shape	Correction	Cluster Shape	Correction	Cluster Shape	Correction
oo	K = 1.12	oooo	K = 1.32	oo oo	K = 1.17
ooo	K = 1.27	ooooo	K = 1.45	o o o o o	K = 1.19
o o o	K = 1.16	oooooo	K = 1.57	oo oo oo	K = 1.17
oooooo o o	K = 1.64	ooooooo	K = 1.73		

Ellipsoidal Particles

For particles that are ellipsoids of revolution, the drag force is given by

$$F_D = 6\pi\mu UaK' \quad (17)$$

where a is the equatorial semi-axis of the ellipsoids and K' is a shape factor.

For the motion of a prolate ellipsoid along the polar axis as shown in Figure 6a,

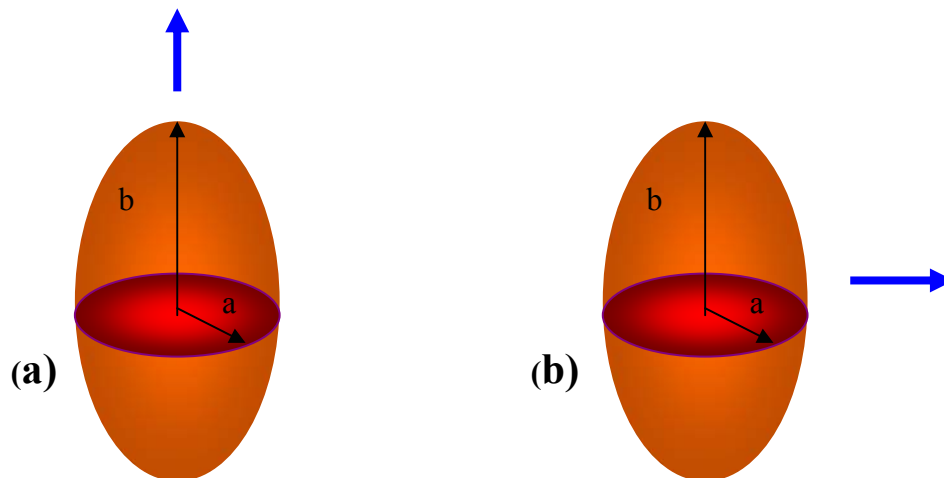


Figure 6. Motions of prolate ellipsoids in a viscous fluid.

$$K' = \frac{\frac{4}{3}(\beta^2 - 1)}{\frac{(2\beta^2 - 1)}{(\beta^2 - 1)^{1/2}} \ln[\beta + (\beta^2 - 1)^{1/2}] - \beta} \quad \left(\beta = \frac{b}{a}\right) \quad (18)$$

where β is the ratio of the major axis b to the minor axis a .

For the motion of a prolate ellipsoid of revolution transverse to the polar axis, as shown in Figure 6b

$$K' = \frac{\frac{8}{3}(\beta^2 - 1)}{\frac{(2\beta^2 - 3)}{(\beta^2 - 1)^{1/2}} \ln[\beta + (\beta^2 - 1)^{1/2}] + \beta} \quad \left(\beta = \frac{b}{a}\right) \quad (19)$$

Similarly for the motion of an oblate ellipsoid of revolution along the polar axis as shown in Figure 7a,

$$K' = \frac{\frac{4}{3}(\beta^2 - 1)}{\frac{\beta(\beta^2 - 2)}{(\beta^2 - 1)^{1/2}} \tan^{-1}(\beta^2 - 1)^{1/2} + \beta} \quad \left(\beta = \frac{a}{b}\right) \quad (20)$$

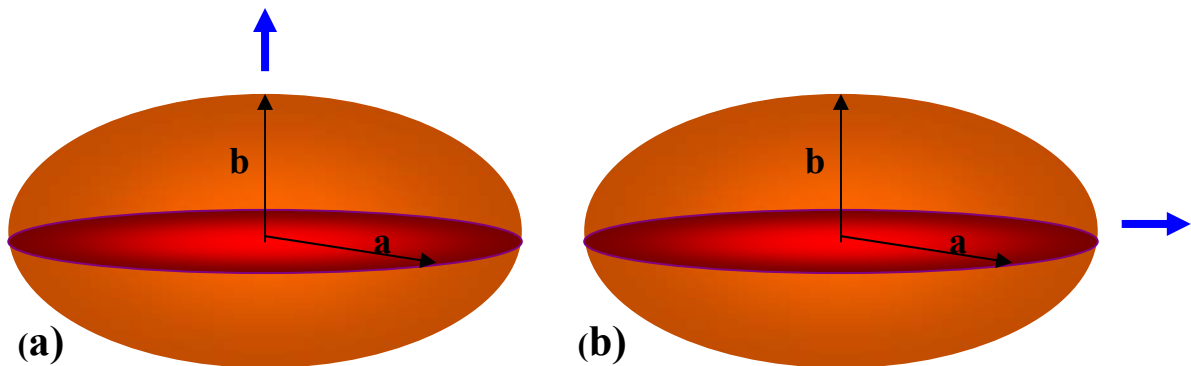


Figure 7. Motions of oblate ellipsoids in a viscous fluid.

For the motion of an oblate ellipsoid transverse to the polar axis as shown in Figure 7b,

$$K' = \frac{\frac{8}{3}(\beta^2 - 1)}{\frac{\beta(3\beta^2 - 2)}{(\beta^2 - 1)^{1/2}} \tan^{-1}(\beta^2 - 1)^{1/2} - \beta} \quad \left(\beta = \frac{a}{b}\right) \quad (21)$$

By taking the limit as $\beta \rightarrow \infty$ in Equations (17)-(21), the drag force on thin disks and needles may be obtained. These are:

Thin Disks of Radius a

For motions perpendicular to the plane of the disk as shown in Figure 8a

$$F_D = 16\mu aU \quad (22)$$

For motions along the plane of the disk as shown in Figure 8b

$$F_D = 32\mu aU/3 \quad (23)$$

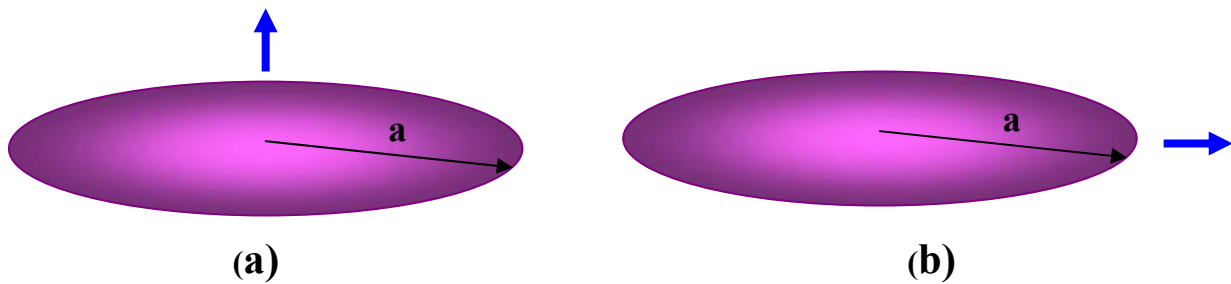


Figure 8. Motions of a thin disk in a viscous fluid.

Ellipsoidal Needle of Length 2b

For motions along the needle as shown in Figure 9a

$$F_D = \frac{4\pi\mu Ub}{\ln 2\beta}, \quad \left(\beta = \frac{b}{a}\right) \quad (24)$$

For side way motions of the needle as shown in Figure 9b

$$F_D = \frac{8\pi\mu Ub}{\ln 2\beta} \quad (25)$$

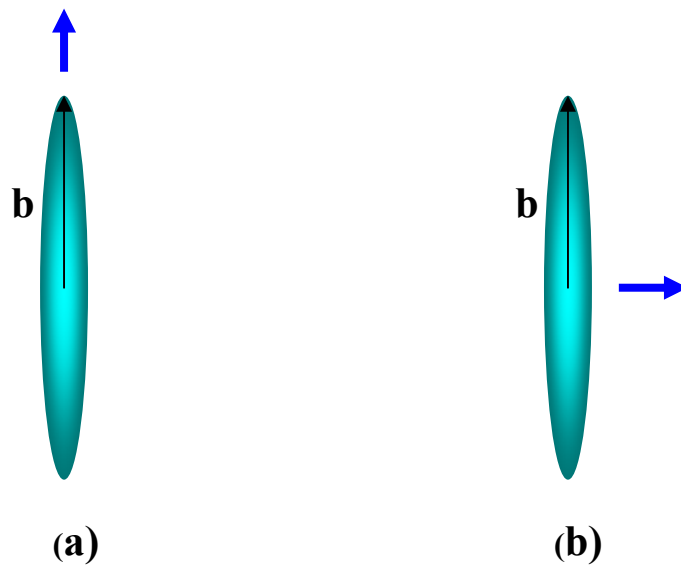


Figure 9. Motions of a needle in a viscous fluid.

Cylindrical Needle

For a cylindrical needle with a very large ratio of length to radius ratio, moving transverse to its axis as shown in Figure 10, the drag per unit length is given as

$$F_D = \frac{4\pi\mu U}{(2.002 - \ln R_e)} \quad (26)$$

where $R_e = \frac{2aU}{\nu}$ and a is the radius. It is understood that



Figure 10. Flow around a cylindrical needle.

Particle Shape Factor

The ratio of the resistance of a given particle to that of a spherical particle having the same volume is called the dynamic shape factor of the particle, K . The radius of an equal volume sphere is referred to as the equivalent radius r_e . Clearly

$$r_e = \alpha\beta^{1/3} \text{ for prolate spheroids,} \quad (27)$$

$$r_e = \alpha\beta^{-1/3} \text{ for oblate spheroids.} \quad (28)$$

Hence,

$$K = K'\beta^{1/3} \text{ for prolate ellipsoids,} \quad (29)$$

$$K = K'\beta^{-1/3} \text{ for oblate ellipsoids.} \quad (30)$$

The Stokes (sedimentation radius) of a particle is the radius of a sphere with the same density, which is settling with the terminal velocity of the particle in a quiescent fluid. Values of shaped factors for a number of particles are available (Hidy, 1984; Lerman, 1979).