INTRODUCTION TO SYMPLECTIC TOPOLOGY

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A symplectic vector space is a pair (V, ω) consisting of finite dimensional real vector space V and a non-degenerate, skew symmetric bilinear form $\omega : V \times V \to \mathbb{R}$, that is
skew symmetry

$$\forall_{v,w \in V}\ \omega(v, w) = -\omega(w, v)$$

non-degeneracy

$$\forall_{v \in V}\ \left(\forall_{w \in V}\ \omega(v, w) = 0 \Rightarrow v = 0\right)$$

Fact: The vector space V is necessary of even dimension.

Linear map $F : (V_1, \omega_1) \to (V_2, \omega_2)$ is called symplectic if

$$F^* \omega_2 = \omega_1,$$

where $F^* \omega_2 (v, w) = \omega_2 (Fv, Fw)$.

Example:
$V = \mathbb{R}^{2n}$, $\omega(x,y) = x^T J_0\, y$, where

$$J_0 = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

That is

$$\omega\left((x_1, ..., x_{2n})^T, (y_1, ..., y_{2n})^T\right) =$$

$$= \sum_{i=1}^{n} (y_i\, x_{n+i} \, - \, x_i\, y_{n+i}).$$

Moreover, this is essentialy the only example of a symplectic vector space. Precisely: if (V, ω) is symplectic , then we can always find a cannonical basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that:

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0$$

$$\omega(e_i, f_j) = \delta_{ij}.$$

Hence two symplectic vector spaces of the same dimension are isomorphic.

Let matrix A represtent linear map

$$A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}.$$

Map A is symplectic if and only if

$$A^T J_0 A = J_0.$$

Matrices satisfying condition above are called symplectic.

Exercise:

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A, B, C, D - real $n \times n$ matrices
Prove that Ψ is symplectic iff

$$\Psi^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$$

More explicitly it means $A^T C$, $B^T D$ are symmetric and $A^T D - C^T B = I$.

Let M be C^∞ smooth manifold, without boundary, compact.
M is a symplectic manifold if there exist on M closed, non-degenerate 2-form ω (called symplectic structure).

Diffeomorphism $\psi : (M_1, \omega_1) \to (M_2, \omega_2)$ is called symplectomorphism if $\psi^* \omega_2 = \omega_1$.

Example:
$M = \mathbb{R}^{2n}$ with coordinates $p_1, \ldots, p_n, q_1, \ldots, q_n$, and

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Note that
$\omega_0((x_1, ..., x_{2n}), (y_1, ..., y_{2n})) =$
$\sum_{i=1}^{n}(x_i y_{n+i} - y_i x_{n+i}) = - < x, J_0 \, y > .$

Fact: Diffeomorphism $\psi : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ is a symplectomorphism if and only if its Jacobi matrix $d\psi$ is a symplectic matrix.

Theorem 1 *(Eliashberg) Group of symplectomorphisms*

$$Symp(M, \omega) = \{g : M \to M \,|g^*\omega = \omega\}$$

is C^0-closed, that is if $g_i \in Symp(M, \omega)$ and $g_i \to g_\infty$ unifromly, then $g_\infty \in Symp(M, \omega)$.

Theorem 2 *(Darboux) For any point y on a symplectic manifold (M^{2n}, ω) of dimension $2n$, there exist an open neighborhood U of y and a differentiable map $f : (U, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ such that $f^*\omega_0 = \omega_{|U}$.*

Denote by $B^{2n}(r)$ the closed Euclidean ball in $\mathbb{R}^{2n}$ with centre 0 and radius r and by

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2}$$

the symplectic cylinder.

Theorem 3 *(Gromov's Nonsqueezing theorem) If there is a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ then $r \leq R$.*

For open subset U of a symplectic manifold (M, ω) define Gromov's capacity

$$c(U) = max\,\{\pi r^2 \,|\, \exists\ B^{2n}(r) \hookrightarrow U \text{ symplectic}\}.$$

Theorem 4 *Any diffeomorphism that preserves capacity i.e. $c(g(U)) = c(U)$ for all open U is such that $g^*\omega = \omega$.*

Example:
S^4 does not admit a symplectic structure.

Assume ω is a closed and non-degenerate 2-form on S^4. As the second de Rham cohomology group of S^4 vanishes, ω has to be exact, that is there exist a 1-form α such that $d\alpha = \omega$. Then also the volume $\Omega = \omega \wedge \omega$ form is exact:
$d(\omega \wedge \alpha) = d\omega \wedge \alpha + \omega \wedge d\alpha = \omega \wedge \omega = \Omega$.
Thus by Stoke's theorem we have

$$\int_{S^4} \Omega = \int_{\partial S^4} \omega \wedge \alpha = 0,$$

which is impossible for a volume form. So we see that on S^4 we cannot impose a symplectic form.