Computable Aspects of Inner Functions

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Outline

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1. Background from analysis
   - The class $H^\infty(\mathbb{D})$
   - Some types of functions in $H^\infty(\mathbb{D})$
   - Some types of inner functions
   - Factorization
   - Frostman's Theorem
2. Background from computability theory
   - Computability over the natural numbers
   - Computability over uncountable spaces: Type-Two Effectivity Theory
3. Statement of results
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The class $H^\infty(\mathbb{D})$

- $\mathbb{D} =_{df} \{z \in \mathbb{C} \ : \ |z| < 1\}$
- $H^\infty(\mathbb{D})$ is the set of all bounded analytic functions $f : \mathbb{D} \to \mathbb{C}$.
- For $f \in H^\infty(\mathbb{D})$, let

$$\| f \|_\infty = \sup\{|f(z)| \ : \ z \in \mathbb{D}\}.$$

- $H^\infty(\mathbb{D})$ is a Banach space under $\| \ \|_\infty$.

Outline

Kinds of functions in $H^\infty(\mathbb{D})$

- $Q \in H^\infty(\mathbb{D})$ is *outer* if there is a positive measurable $\phi : \partial\mathbb{D} \to \mathbb{R}$ such that $\log \phi \in L^1(\partial\mathbb{D})$ and

$$Q(z) = \lambda \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \phi(e^{it}) dt \right\}.$$

  for some $\lambda \in \partial\mathbb{D}$.
- $u \in H^\infty(\mathbb{D})$ is *inner* if $\lim_{z \to z_0} |u(z)| = 1$ for almost all $z_0 \in \partial\mathbb{D}$.

Outline

Singular functions

Definition

A function $s \in H^{\infty}(\mathbb{D})$ is *singular* if there is a finite positive Borel measure on $\partial\mathbb{D}$, μ, that is singular with respect to Lebesgue measure and such that

$$s(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right\}$$

Theorem

If s is singular, then:

1. *s is inner.*
2. $s(0)$ *is a positive real number.*
3. *s has no zeros.*

Blaschke products

Definition

Let $A = \{a_n\}_{n=0}^\infty$ be a sequence of points in $\mathbb{D} - \{0\}$. The product

$$B_{A,k}(z) =_{df} z^k \prod_{n=0}^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a *Blaschke product*. We abbreviate $B_{A,0}$ with B_A.

Definition

Let $A = \{a_n\}_{n=0}^{\infty}$ be a sequence of points in $\mathbb{D} - \{0\}$. The series

$$\Sigma_A =_{df} \sum_{n=0}^{\infty}(1 - |a_n|)$$

is called the Blaschke sum of A. The inequality

$$\sum_{n=0}^{\infty}(1 - |a_n|) < \infty$$

is called the *Blaschke condition*.

Theorem

Let $A = \{a_n\}_{n=0}^\infty$ be a sequence of points in $\mathbb{D} - \{0\}$.

1. *If A satisfies the Blaschke condition, then $B_{A,k}$ is an inner function.*
2. *If A satisfies the Blaschke condition, then the terms of A are precisely the zeros of B_A. Furthermore, the number of times a zero of B_A appears in A is its multiplicity.*
3. *If A does not satisfy the Blaschke condition, then $B_A \equiv 0$.*

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Definition

N is the class of all $f \in H^\infty(\mathbb{D})$ such that

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta < \infty$$

Theorem

(Canonical Factorization Theorem) *If $f \in N$, then there exist λ, F, B, S_1, and S_2 such that*

$$f(z) = \lambda F(z) B(z) \frac{S_1(z)}{S_2(z)}$$

where $\lambda \in \partial\mathbb{D}$, B is a (possibly finite) Blaschke product, and S_1, S_2 are singular functions.

Corollary

(Factorization of Inner Functions) *If u is an inner function, then there exist unique λ_u, b_u, s_u such that $u = \lambda_u b_u s_u$, $\lambda_u \in \partial\mathbb{D}$, b_u is a (possibly finite) Blaschke product, and s_u is a singular function.*

Outline

For each closed $K \subseteq \mathbb{D}$ and each positive measure σ on K, let $U_\sigma : \mathbb{D} \to \mathbb{D}$ be defined by the equation

$$U_\sigma(z) = \int_K \log \frac{1}{|z - \zeta|} d\sigma(\zeta).$$

Definition

Let $F \subseteq \mathbb{D}$ be closed. We say that F has *zero capacity* if for every positive measure on F, σ, with $\sigma \neq 0$, U_σ is not bounded on any neighborhood of F. Otherwise, we say that F has *positive capacity*. If U is an arbitrary subset of $\mathbb{D}$, then we say that U has positive capacity just in case it has a closed subset with positive capacity; otherwise, we say that it has zero capacity.

Facts about capacity

Theorem

Every zero-capacity set has measure zero.

The Cantor set has *positive* capacity.

For $a, z \in \overline{\mathbb{D}}$ with $|a| < 1$, let

$$M_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Theorem

(Frostman's Theorem) *Let u be a non-constant inner function. Then, $M_a \circ u$ is a unit multiple of a Blaschke product for all $a \in \mathbb{D}$ except in a set of capacity zero.*

The set of values of a for which $M_a \circ u$ is not a unit multiple of a Blaschke product is called the *exception set of u*.

Corollary

If u is a non-constant inner function, and if $\epsilon > 0$, then there is a unit multiple of a Blaschke product B such that $\| u - B \|_{\infty} < \epsilon$.

Some questions

1. Given A, can one "compute" B_A?
2. Given an inner function u, can one "compute" its factorization?
3. Given an inner function u and a number $\epsilon > 0$, can one "compute" a unit multiple of a Blaschke product B such that $\| u - B \|_\infty < \epsilon$.

Outline

Fix a finite alphabet Σ with $0, 1 \in \Sigma$.

Let Σ^* be the set of all finite sequences whose terms are all in Σ.

Let $f :\subseteq A \rightarrow B$ denote that $dom(f) \subseteq A$ and $ran(f) \subseteq B$.

Turing machines

Definition

A function $f :\subseteq \Sigma^* \to \Sigma^*$ is *computable* if it can be computed by a Turing machine. Meaning:

1. If input string σ is not in domain of f, then machine does not halt on input σ.
2. If input string σ is in domain of f, then machine eventually halts and $f(\sigma)$ is written on tape.

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Two fundamental ideas:

- Representations
- Type-two machines

Some notation:

- Let Σ^ω be the set of all *infinite* sequences whose terms are all in Σ.
- Let $\iota(a_0, a_1, \ldots, a_n) = 110a_00a_10...a_n011$.
- $w \lhd p$ denote that p can be written in the form $p = uwv$ for some $u \in \Sigma^*$ and $v \in \Sigma^\omega$.

Definition

Let M be a set. A *representation of* M is a surjective function $\delta :\subseteq \Sigma^\omega \rightarrow M$.

Representations are also called *naming systems*.

If $\delta(p) = x$, then we say that p is a *δ-name of x*.

Definition

$x \in M$ is *δ-computable if it has a computable δ-name*.

A recipe for representations

1. Start with a second countable T_0 space (M, σ) (σ a countable subbasis).
2. Assume you have surjective $\nu : \Sigma^* \to \sigma$ such that $\{(w, w') \mid \nu(w) = \nu(w')\}$ is computable. Define $\mathcal{S} = (M, \sigma, \nu)$.
3. For each $p \in \Sigma^\omega$, let $\delta_{\mathcal{S}}(p)$ be the $x \in M$ (if there is one) such that

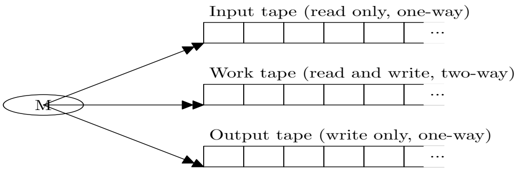
$$\iota(w) \lhd p \;\Leftrightarrow\; x \in \nu(w)$$

for all $w \in \Sigma^*$.
(The idea is that $\delta_{\mathcal{S}}(p) = x$ iff p "encodes an enumeration" of all subbasic neighborhoods that contain x.)

Some useful representations

- ρ^2. A representation of $\mathbb{C}$. Start with standard basis for $\mathbb{C}$.
- δ_{CO}. A representation of $C(\mathbb{C})$. Start with compact-open topology on $\mathbb{C}$.
- $[\rho^2]^\omega$. A representation of set of all infinite sequences of complex numbers. Use product topology.
- Given $\mathcal{S}_1$ and $\mathcal{S}_2$, let $[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}]$ be the representation given by starting out with the product topology of $\mathcal{S}_1$ and $\mathcal{S}_2$. Define $[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}, \delta_{\mathcal{S}_3}] = [[\delta_{\mathcal{S}_1}, \delta_{\mathcal{S}_2}], \delta_{\mathcal{S}_3}]$. *etc.*

Type-two machines



Computable functions

Definition

Let $f :\subseteq \Sigma^\omega \to \Sigma^\omega$. We say that f is *computable* if there is a type-two machine M such that for every $p \in \Sigma^\omega$, when p is written on the input tape and M is allowed to run, then:

- If $p \in dom(f)$, then M writes $f(p)$ on the output tape.
- If $p \notin dom(f)$, then M writes only finitely many symbols on the output tape.

Definition

Let $\delta_i :\subseteq \Sigma^\omega \to M_i$ be a representation of M_i for $i = 0, 1$. Let $f : M_0 \to M_1$. Then, f is (δ_0, δ_1)*-computable* if there exists computable $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta_1 F(p) = f\delta_0(p)$ for all $p \in dom(\delta_0)$.

Theorem

(Matheson, McNicholl, 2006) *There is a* $[\rho^2]^\omega$*-computable sequence* $A = \{a_n\}_{n=0}^{\infty}$ *such that* B_A *is not* (ρ^2, ρ^2)*-computable.*

In other words, merely knowing the Blaschke sequence is not enough to compute the Blaschke product.

Theorem

(Matheson, McNicholl, 2006) *If* B_A *is* (ρ^2, ρ^2)*-computable, then* A *is* $[\rho^2]^\omega$ *computable.*

Theorem

(McNicholl, 2007) *The map* $(A, \sum_A) \mapsto B_A$ *is* $([[\rho^2]^\omega, \rho^2], \delta_{CO})$*-computable.*

In other words, if you know a Blaschke sequence and its Blaschke sum, then you can compute the Blaschke product.

Theorem

(McNicholl, 2007) *The map* $(A, B_A) \mapsto \sum_A$ *is* $([[\rho^2]^\omega, \delta_{CO}], \rho^2)$*-computable. In fact,* $(A, B_A(0)) \mapsto \sum_A$ *is* $([[\rho^2]^\omega, \rho^2], \rho^2)$*-computable.*

In other words, once you know a Blaschke sequence, in order to compute the Blaschke product you have to know the Blaschke sum (or an equivalent piece of information).

Corollary

(McNicholl 2007) *Suppose A is* $[\rho^2]^\omega$*-computable. If* B_A *maps* ρ^2*-computable complex numbers to* ρ^2*-computable complex numbers, then* B_A *is* (ρ^2, ρ^2)*-computable.*

This is not the case for power series!

Theorem

(McNicholl, 2007) *There is a* $([\delta_{CO}, \rho^2], \rho^2)$*-computable function* Ψ *such that if* u *is inner and* $\epsilon > 0$*, then* $M_{\Psi(u,\epsilon)}$ *is a Blaschke product and* $\| u - M_{\Psi(u,\epsilon)} \|_\infty < \epsilon$*.*

Theorem

(McNicholl, 2007) *The map* $u \mapsto (\lambda_u, b_u, s_u)$ *is not* $(\delta_{CO}, [\rho^2, \delta_{CO}, \delta_{CO}])$*-computable.*

In other words, merely knowing an inner function is not enough to compute its factorization.

Let $\sum_u$ denote $\sum_{n=0}^{\infty}(1 - |z_n|)$ where $z_0, z_1, \ldots$ are the non-zero zeros of u. Let k_u denote the order of u's zero at 0 if there is one; if $u(0) \neq 0$, then let $k_u = 0$.

Theorem

(McNicholl, 2007) *The map* $(u, \sum_u, k_u) \mapsto (\lambda_u, b_u, s_u)$ *is* $([\delta_{CO}, \rho^2, \rho^2], [\rho^2, \delta_{CO}, \delta_{CO}])$*-computable. (Provided u has infinitely many zeros.)*

Theorem

(McNicholl, 2007) *The map* $(u, k_u, b_u) \mapsto \sum_u$ *is* $([\delta_{CO}, \rho^2, \delta_{CO}], \rho^2)$*-computable. (Provided u has infinitely many zeros.)*

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