An application of algorithmic information theory

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Preliminaries

- $A \leq_T B$ if there is an algorithm using B as an oracle that will compute the characteristic function of A.
- $A \leq_{wtt} B$ if there's an algorithm like before, but also a computable function that limits how much of the oracle B the algorithm can use.
- The *Turing degree of the set* A, $deg(A)$ is the collection of all sets $\equiv_T$ to A.
- The *wtt-degree of the set* A, $deg_{wtt}(A)$ is the collection of all sets $\equiv_{wtt}$ to A.

Background

We consider computable linear orderings (CLOs) $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$, and think about an additional relation R on the structure.

Example

$\mathcal{L} \cong \omega + \omega^*$ with additional relation $R = \omega_{\mathcal{L}}$.

$$\bullet \; \bullet \; \bullet \; \ldots \quad \ldots \; \bullet \; \bullet \; \bullet$$

- The *degree spectrum of relation R on a computable structure $\mathcal{M}$*, $DgSp_{\mathcal{M}}(R)$, is the collection of all Turing degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.
- The *wtt-spectrum of relation R on a computable structure $\mathcal{M}$*, $DgSp^{wtt}_{\mathcal{M}}(R)$, is the collection of all wtt-degrees of images of R in computable structures $\mathcal{N} \cong \mathcal{M}$.

Context and some facts about $\omega + \omega^*$

Let $\mathcal{L}$ be a CLO isomorphic to $\omega + \omega^*$, and $\omega_{\mathcal{L}}$ the ω-part of $\mathcal{L}$.

- (Harizanov, 1998) The (Turing) degree spectrum of $\omega_{\mathcal{L}}$ is exactly the Δ^0_2-degrees.
- Is the same true of the wtt-spectrum? Does it consist of all wtt-degrees that are wtt-computable from the halting set? No.

This is what we *can* say:

Theorem
For every Δ^0_2 set A, there is a CLO $\mathcal{L}$ of order type $\omega + \omega^*$ with $A \leq_T \omega_{\mathcal{L}} \leq_{wtt} A$.

We'll see that this is the best we can do: $\leq_T$ can't be replaced with $\leq_{wtt}$ in the Theorem.

A much stronger statement

Theorem

There is a c.e. set D that is not wtt-reducible to any initial segment of any computable scattered linear ordering.

(A linear ordering is *scattered* just in case it fails to contain a copy of $\mathcal{Q} = \langle \mathbb{Q}, <_{\mathbb{Q}} \rangle$. For example, $\omega + \omega^*$.)

The punchline: The halting set, $0'$, itself will be this set.

We will see that if $0'$ is wtt-reducible to an initial segment of a CLO, then that linear ordering is not scattered.

Though $0'$ is at the top of the Δ^0_2 sets, we can find a *low* c.e. set that does the same thing.

A nice fact about scattered linear orderings

Let $\mathcal{L}$ be a countable linear ordering. Then $\mathcal{L}$ is scattered iff $\mathcal{L}$ has only countably many initial segments.

If $\mathcal{L}$ is a CLO, then $\mathcal{L}$ is scattered iff each of its initial segments is *ranked* – an element of a countable Π^0_1 class.

(A set of sets of natural numbers is a Π^0_1 class if it is the collection of paths through a computable tree.)

Fact

Let $\mathcal{L}$ be a countable linear ordering. Then $\mathcal{L}$ is scattered iff $\mathcal{L}$ has only countably many initial segments.

Proof. $\leftarrow$. If $\mathcal{L}$ has a copy of $\mathcal{Q}$, it has as many initial segments as $\mathcal{Q}$ does... uncountably many.

$\rightarrow$. Suppose $\mathcal{L}$ has uncountably many initial segments... then it has a copy of $\mathcal{Q}$:

- Let $\mathcal{I}$ be the collection of initial segments of $\mathcal{L}$ (view these as paths through a subtree of $2^{<\omega}$).
- $\mathcal{I}$ is a closed uncountable set in Cantor space 2^ω, and so has a perfect subset $\mathcal{J}$. Take T to be the perfect subtree of $2^{<\omega}$ with $[T] = \mathcal{J}$.
- For each branching node of T, take a_σ to be an element of $\mathcal{L}$ that the extending nodes disagree on.
- It's easy to check that these a_σ's form a copy of $\mathcal{Q}$.

So, we need to show that if an initial segment of a CLO wtt-computes $0'$, then that CLO has uncountably many initial segments.

Equivalently, the collection of initial segments has a (nonempty) perfect subset.

To do this, we'll use facts about Π^0_1 classes and their members since the collection of initial segments forms a Π^0_1 class.

Some definitions.

- For finite strings σ, the *Kolmogorov complexity* of σ, $C(\sigma)$, is the length of the shortest program you can write that will output σ.
- An *order* is a computable, nondecreasing, unbounded function.
- A set A is *complex* if there is an order g so that

$$\forall n \; C(A \upharpoonright n) \geq g(n).$$

- A function f is *diagonally non-computable* (DNC) if for each $e \in \omega$, the value of $f(e)$ is different from $\varphi_e(e)$ whenever $\varphi_e(e) \downarrow$.

Some facts.

Theorem (Kjos-Hanssen, Merkle and Stephan)
A set A is complex iff there is a DNC function $f \leq_{wtt} A$.

So...

- If $A \leq_{wtt} B$ and A is complex, so is B. ($\leq_{wtt}$ is transitive.)
- $0'$ is complex. Why? $0'$ wtt-computes

$$f(e) = \begin{cases} \varphi_e(e) + 1 & \text{if } \varphi_e(e) \downarrow \\ 0 & \text{if } \varphi_e(e) \uparrow \end{cases} .$$

A theorem about Π^0_1 classes

Theorem
Let P be a Π^0_1 class with a complex element A. Then P has a perfect Π^0_1 subclass Q with $A \in Q$.

Proof. Let g be an order witnessing that A is complex:

$$\forall n\, C(A \restriction n) \geq g(n).$$

Set $Q = \{X \in P | \forall n\, C(X \restriction n) \geq g(n)\}$, and note that Q is a Π^0_1 subclass of P and that it is nonempty. (A is in it.)

By definition, every element in Q is complex, and so can't have any isolated elements (they would be computable!), so Q has to be perfect.

$0'$ is not wtt-reducible to any initial segment of any scattered CLO

Take a CLO $\mathcal{L}$ with an initial segment A that wtt-computes $0'$. Let P be the (Π^0_1) class of initial segments of $\mathcal{L}$.

A is complex since $0'$ is, and is an element of P, so P has a nonempty perfect Π^0_1 subclass by the Theorem we just proved, and so $\mathcal{L}$ must have uncountably many initial segments.

By the earlier lemma, we see that $\mathcal{L}$ contains a copy of the rationals, and so is not scattered.

References

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