OESTERREICHISCHE NATIONALBANK

SYSTEMIC RISK MONITOR

Risk Assessment and Stress Testing for the Austrian Banking System

Model Documentation

Contents

Preface

This book documents the main ideas and concepts of *Systemic Risk Monitor* (SRM), a model for systemic financial stability analysis and stress testing of the Austrian banking system. These ideas constitute the foundations of SRM and are results from a research program on banking regulation and systemic risk undertaken at the research department of Oesterreichische Nationalbank. Based on these results, the model was developed jointly by the OeNB research department and financial stability division in collaboration with the University of Vienna and Fachhochschule Vorarlberg. The output of the project is a software that allows quantitative systemic stability analysis and stress testing for analysts of the financial stability division of OeNB. Results are shared with banking supervisors at OeNB and the Austrian Financial service Authority (FMA). This book is focused on the main concepts and ideas. Application handbooks for the software itself are written down in separate documents.

We have made an attempt to present the model at various levels of detail so that it is useful both for readers who want a quick overview as well as for readers that are interested in open research questions or technical modeling details to implement their own systemic risk analysis model. A quick preview of the different chapters can be given as follows:

Chapter 1 This is an introductory chapter. It gives a general overview and shows some applications to illustrate the analysis usually undertaken by SRM with a typical data set.

Chapter 2 This chapter explains how SRM models risks affecting the banking system. For this purpose a multivariate distribution of risk factor changes is modeled by a two step procedure. In the first step models for the marginal distributions are selected based on optimal out of sample density forecasting criteria. In a second step dependency is captured by a grouped t-copula.

Chapter 3 This chapter gives an introduction to the modeling of market risk in SRM. It is short because in this respect SRM relies on traditional techniques well established in quantitative risk management.

Chapter 4 This chapter givens an introduction to the modeling of credit risk with counterparties outside of the banking system, mostly corporations and private households. While the credit risk model is embedded in established techniques known from the literature the modelling of systematic risk that drives default correlation differs from traditional models. In SRM the credit risk model is related to the multivariate distribution of risk factor changes by a statistically estimated relation between macroeconomic risk factors and default rates in various industry sectors.

Chapter 5 This chapter gives an introduction to the core element of SRM which takes traditional market- and credit risk analysis to the system level by combining it with a network model of interbank relations. This is the core innovation of SRM that allows an integrated analysis of different risk categories and different data on the banking system as a whole.

Chapter 6 This chapter gives an overview of the data used by SRM. In the implementation of the model the organisation of the data to make them useful and usable as a regular input was a major practical challenge. This chapter might be therefore be of particular interest to readers at other institutions that are in charge of financial stability of supervision and consider to use similar models.

Chapter 7 This chapter gives a conceptual overview of the implementation of the model. It conveys the main ideas and principles of the organisation of the software.

Appendix The appendix gives a technical discussion of the model. This might be of interest for researchers who want to see the ultimate details of the model.

To get a broad idea and a quick overview of the general principles of SRM and how they are applied we hope that the reader gets a fairly clear picture from reading the introductory chapter 1. A reader who is happy with the broad overview of the concepts given in chapter 1 might be interested to learn more about the practical matters such as data and implementation. In this case it might be interesting to combine chapter 1 with chapter 6 and 7. An overview of the different model parts in greater detail can be gained from reading chapters 2 to 5. Reading this part of the book will be most suitable for the reader who wants to get an idea of the main concepts and some information about the technical details of the different parts of the model without going into all of the technical details and derivations of the ideas. Finally a reader who is technically interested in the model is best advised to start by chapter 1 and then go directly to the appendix.

The work on the project has left us indebted to a number of people who supported us with resources, advice and hands on help in many practical aspects of the work. We would first of all like to thank Oesterreichische Nationalbank for supporting our project that had an unusually long time horizon. We benefited very much from the support of Andreas Ittner, Eduard Hochreiter and Michael Wuerz. Among the many colleagues who helped us with their expert advice and direct support we would like to mention particularly Martin Jandacka, Wolfgang Wegschaider, Simon Flöry and Christine Zulehner.

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Chapter 1

Introduction to Systemic Risk Monitor

1.1 Systemic Financial Stability Analysis

The ultimate mandate of central banks is to achieve and maintain price stability. Safeguarding and maintaining financial stability has always been regarded as a necessary prerequisite for this task. Institutionally this combination of tasks was until very recently achieved by putting the central bank in charge of the oversight of individual financial institutions. Following the lead of the UK, many countries, including Austria, have transferred responsibility for the oversight of individual financial institutions to new financial supervision authorities, while the central banks kept the mandate to safeguard and maintain systemic financial stability. These institutional developments have forced central banks to arrive at answers to the new question what it means to maintain systemic financial stability without having ultimate responsibility for the oversight over individual financial institutions.

Beginning in 2002 the Oesterreichische Nationalbank (OeNB) launched in parallel several projects that aim to develop modern tools for systemic financial stability analysis and off-site banking supervision. In these projects OeNB expertise from financial analysis and research was combined with expertise from the University of Vienna, the University of Applied Sciences Vorarlberg, the University of Technology Vienna and the Austrian Financial Service Authority (FMA) (see OeNB and FMA (2005)).

Systemic Risk Monitor (SRM) is part of this effort. SRM is a model to analyze banking supervision data and the major loan register collected at OeNB in an integrated quantitative risk management model. The purpose of SRM is to assess at a quarterly frequency systemic risk - the probability of a major breakdown in financial intermediation - in the Austrian banking system. SRM is also used to perform regular stress testing exercises.

1.2 An Overview of the SRM Model

The basic idea of the SRM model is to combine standard techniques from modern quantitative market- and credit risk management with a network model of the banking system. In contrast to standard risk management models, SRM makes the step from the individual institution perspective to the system level. This step is the major challenge to be met by any systemic risk model. Only at the system level the two major reasons for simultaneous defaults become visible: Correlated exposures and financial inter-linkages. The risk of simultaneous defaults of institutions and the financial losses of such events is the key focus of systemic financial stability analysis. SRM can draw on a rich modern literature dealing with risk management and risk monitoring problems for banks or insurance companies. The change of perspective from the individual institution level to the system level is the main methodological innovation of SRM. It is this system perspective where SRM had to explore new territory. This chapter gives an overview of the main concepts and ideas used in our approach as well as a brief overview of how these concepts are applied for systemic financial stability analysis. We will follow closely the excellent discussion in Chapter 2 of McNeil et al. [2005].

1.2.1 Some concepts from quantitative risk management

SRM describes the Austrian banking system at the beginning of each quarter as a system of portfolios. Each portfolio in the system belongs to one bank and typically consists of collections of securities such as stocks and bonds across domestic and foreign markets, a collection of corporate loans as well as other assets and liabilities such as deposits and interbank loans. The value of a portfolio of a particular bank i at time s, called the observation time, is denoted by $V_i(s)$ and the vector of portfolio values in the entire banking system is denoted accordingly by $\mathbf{V}(s) = (V_1(s), \ldots, V_n(s))$. Portfolio values are random variables and it is assumed that at time s the system of portfolio values $\mathbf{V}(s)$ is observable from the data available at OeNB.

For a given time horizon h, which in SRM is always one quarter (approximately 60 trading days) the loss or gain of the system of portfolios over the period $[s, s+h]$ is given by the vector

$$\mathbf{L}_{[s,s+h]} = (\mathbf{V}(s+h) - \mathbf{V}(s)) \tag{1.1}$$

While $\mathbf{L}_{[s,s+h]}$ is observable at $s+h$ it is random from the viewpoint of the observation time s. The ultimate goal of SRM is to model the distribution of $\mathbf{L}_{[s,s+h]}$ over a quarterly horizon. All questions concerning systemic risk, the probability of simultaneous failures and the losses associated with these events are answered with respect to this distribution.

We adopt the usual practice from risk management to think of future portfolio values as a function of time t as well as a d-dimensional vector of risk factors $\mathbf{Z}_t = (Z_{t,1}, \ldots, Z_{t,d})$

$$\mathbf{V}_t = \mathbf{f}(t, \mathbf{Z}_t) \tag{1.2}$$

$\mathbf{f}(t, \mathbf{Z}_t)$ is the vector $(f_1(t, \mathbf{Z}_t), \ldots f_n(t, \mathbf{Z}_t))$ consisting of the individual functions f_i of each bank. It is assumed that the risk factors are observable and known at the observation time. In risk analysis we are usually interested in risk factor changes over the analysis horizon. Denoting the vector of risk factor changes by $\mathbf{X}_t$ the loss of the system of portfolios can be written as

$$\mathbf{L}_{t+1} = (\mathbf{f}(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - \mathbf{f}(t, \mathbf{Z}_t)) \tag{1.3}$$

All individual modeling steps as well as the practical challenges that arise in SRM have to do with the details of how we describe the system of functions mapping risk factor changes to portfolio losses.

From this discussion we see a fundamental modelling choice taken in SRM. Following the literature on risk management of individual institutions the analysis is undertaken for a given system of portfolios observed at the observation time. The value of the portfolios is assumed to be completely determined by the risk factors and all behavioral considerations are not taken into account. The longer the time horizon under consideration the more problematic is such an assumption. In particular in our framework where we aim at an integrated analysis of portfolio positions who can be turned over easily with others that are much more difficult to change even at a 60 day horizon, this assumption is debatable for some of the portfolio positions. We ask the following question: given the portfolio positions we observe today in the system and given the potential future realizations of risk factors, how would these changes influence portfolio values ceteris paribus? This allows a statement about the risk inherent in the current system of portfolios.

1.2.2 Mapping risk factors to portfolio positions

In the construction of portfolios for the Austrian banking system we conceptually distinguish three major categories of positions. The first category are positions of marketable securities such as stocks, bonds, or assets and liabilities denominated in Euro or in in foreign exchange. The second category contains all loans with counterparties outside of the banking system, mainly corporations. The third category contains all positions held among the Austrian banks, interbank loans as well as interbank shares. The three categories can be distinguished by the complexity of the function mapping risk factor changes into losses (or gains) of the portfolio values.

For marketable securities the situation is fairly simple. Supervisory data allow us a fairly coarse reconstruction of positions of market values of securities that are held on the bank balance sheet. The picture is coarse because individual stocks are lumped into Austrian and foreign and interest and currency sensitive instruments are mapped into broad maturity and currency buckets. Consider for instance a simple stock portfolio consisting of Austrian and foreign stocks. Risk factor changes are then the logarithmic changes in the Austrian and a foreign stock price index. To calculate gains or losses from the stock portfolios we can use a linearized approximation of the loss function. This amounts then to simply multiplying the position values with the risk factor changes to get the portfolio gains and losses. For interest and currency sensitive positions we can equally arrive at gains and losses by using linearized losses and the relevant risk factor changes, that is changes in different exchange rates, the main international exchange rates with the Euro or interest rate changes for different maturities and different currencies.

For loans to non banks the situation is more complicated because the dependence between loan losses and risk factors is more indirect. We don't have a simple analogue to market returns. Defaults of loans in particular industry sector - the units to which we can break down loans in SRM - are driven by default indicators. The probability distribution of these indicators depends mainly on risk factors describing the aggregate state of the economy, i.e. the driving risk factors are macroeconomic variables. Due to the discrete nature of the default indicators linearized losses are of little importance for the analysis of credit risk. Therefore SRM uses a credit risk model to calculate losses from corporate loan portfolios. The basic idea is that the default probability of a loan in a particular industry sector – say construction – depends on a set of macroeconomic variables according to a function the parameters of which are statistically estimated from historical data. Given a realization of macroeconomic variables and the implied probability of default for different industry sectors, loan defaults are assumed to be conditionally independent. Under this assumption a loan loss distribution can be derived for each bank for each value of macroeconomic risk factor changes. Loan losses are then calculated by independent draws from these loan loss distributions. For loan losses, therefore, the function mapping risk factor changes (changes in macroeconomic variables) to loan losses is more complicated.

Finally gains and losses from interbank positions are calculated by the use of a network clearing model. The basic idea of this model is to capture interbank loans and shares by a matrix of all bilateral positions as observed at the observation time. Risk factor changes that have an impact on the value of loans and market positions together with the network of interbank loans determine for each bank whether it can fulfill its interbank promises or not. If one or more banks are unable to fulfill their interbank obligations for a particular realisation of risk factor changes a clearing procedure redistributes the value of insolvent institutions among the creditors until all financial claims after the realization of risk factor changes become consistent. Thus in the case of interbank positions the function mapping risk factor changes into interbank losses is a fairly complicated function of market and credit losses and the clearing procedure.

All these losses in combination determine

$$\mathbf{L}_{t+1} = (\mathbf{f}(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - \mathbf{f}(t, \mathbf{Z}_t)) \tag{1.4}$$

in SRM.

Whereas the modelling of non-interbank market and credit losses is rooted in standard quantitative risk management techniques the combination with an interbank network model to arrive at total gains and losses in the banking system in SRM are new. Both generalizations of standard individual risk management techniques, the simultaneous consideration of portfolio values across the system for given risk factor changes and the resolution of bilateral claims via an network clearing model, focus on the main issues for an institution in charge of monitoring systemic financial stability, the probability of joint default of institutions and its financial consequences.

1.2.3 Calculating systemic losses

A graphical description of the model structure is given in Figure 1.1. The figure displays the modular construction of SRM. All modules will be described in detail in the following chapters.

At the top of Figure 1.1 is a model of a multivariate risk factor change distribution. This distribution is estimated every quarter based on past observations of market price changes and changes of macroeconomic variables that have an important impact on default probabilities. The modeling strategy treats the marginal risk factor distributions and the dependency structure separately. While marginal distributions are chosen according to statistical tests that select for each risk factor a model which gives the best out of sample density forecast of changes in each risk factor over a three month horizon, the dependence is modeled by fitting a grouped t-copula to the data. Together the marginal distribution and the copula characterize the multivariate risk factor change distribution. For the simulation of scenarios vectors of risk factor changes are drawn at random from this distribution.

Each draw of risk factor changes from the multivariate distribution characterizes a scenario. Scenarios are then translated into profits and losses at the system level by the procedures described above. This is achieved in two steps. In a first step

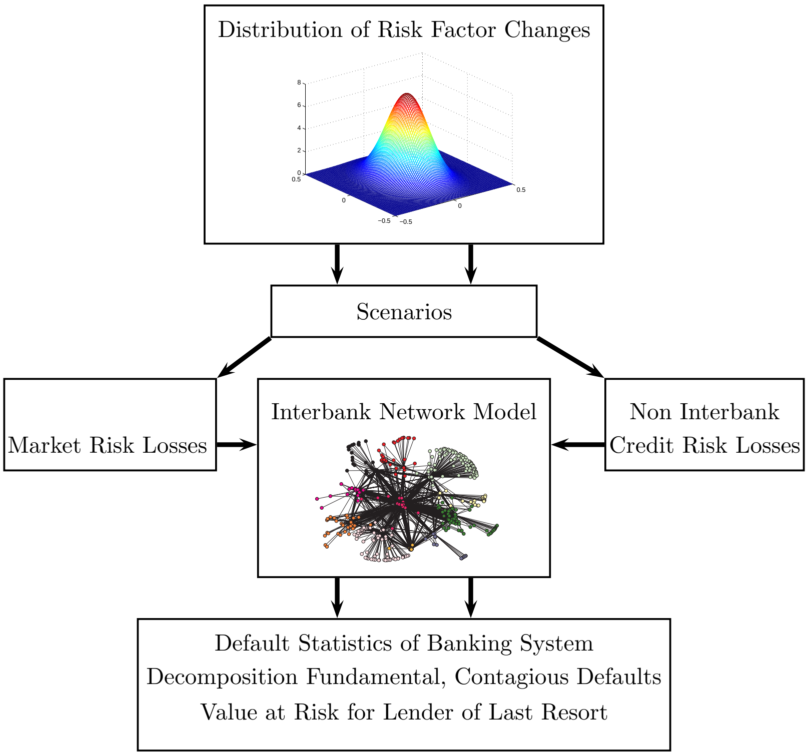

Figure 1.1. *The figure shows the basic structure of the SRM model. Banks non interbank portfolios are exposed to shocks from a risk factor change distribution of market and credit risk factors. The value of inter-bank positions is determined endogenously by the network model and a clearing mechanism that makes all financial claims consistent ex post after shocks have been realized. The clearing of the inter-bank market determines the solvency of other banks and defines endogenous default probabilities for banks as well as the respective recovery rates. The output consists of insolvency statistics, a decomposition into fundamental and contagious defaults and an estimate about the amounts of liquidity a lender of last resort hast to stand ready to inject into the system. The individual loss components are available for the analyses of the overall loss distribution as well as the market-, credit- and interbank risk losses.*

each scenario is analyzed with respect to its impact on the value of non interbank market and credit positions.

These positions are then combined with the network model. The output of the clearing model gives the final result for the banking system for each scenario. Simulating many scenarios we get a distribution of insolvency and gains and losses for the banking system that allows us to make probability assignments for insolvencies over a three month horizon.

We use four main risk concepts to look at the simulation output.

1. Analysis of fundamental and contagious defaults.

2. Analysis of PD distribution according to rating classes.

3. Analysis of aggregate loss distributions.

4. Quantification of resources that might have to be mobilized by a lender of last resort.

Since the risk of bank defaults in particular of joint defaults and the large scale breakdown of intermediation is of major interest to the central bank we put a particular focus on bank defaults and default probabilities. The network model allows us to distinguish default events that directly result from changes in risk factors from defaults that result indirectly from contagion of insolvency through interbank relations. We call defaults fundamental if they result directly from risk factor movements and we call them contagious if they are a consequence of interbank insolvency contagion. Apart from analyzing the number of fundamental and contagious defaults we look at the distribution of the simulated default probabilities according to the OeNB's rating classes. We look at the aggregate loss distribution both for all risk categories in combination and for certain subcomponents such as market risk, credit risk and contagion risk. Finally we make an attempt to quantify the resources a lender of last resort might have to mobilize to prevent insolvencies.

1.2.4 Stress Testing

An advantage of a quantitative model is that it allows the consideration of hypothetical situations. In the context of systemic risk assessment one kind of thought experiments is of particular importance. Usually it is of interest how the risk measures for the banking system will behave under extreme developments of risk factor changes. Such thought experiments are known as stress testing. Systemic risk monitor provides a coherent framework to consistently conduct such stress testing exercises.

In a stress test one or more risk factors of interest are constrained to take extreme values, like a certain drop in GDP, or a hike in short term interest rates. Since we have a complete model of the multivariate risk factor distribution we can then perform a model simulation conditional on the constraint that certain risk factors are at their stressed values. The risk measures of the model can then be studied relative to the baseline simulation based on the unconditional risk factor change distribution calibrated to historical data. The main advantage of this approach is the consistency with the dependence structure of the risk factors and therefore the consistency with the quantitative framework. Such an approach is advocated by Elsinger et al. [2006a] or Bonti et al. [2005].

1.3 Data

The main sources of data used by SRM are bank balance sheet and supervisory data from the monthly reports (MAUS) to the Austrian Central Bank (OeNB) and the database of the OeNB major loans register (Großkreditevidenz, GKE). In addition we use default frequency data in certain industry groups from the Austrian rating agency Kreditschutzverband, financial market price data from Bloomberg, and macroeconomic time series from OeNB, the OECD and the IMF International Financial Statistics.

Banks in Austria file monthly reports on their business activities to the central bank. In addition to balance sheet data, MAUS contains a fairly extensive assortment of other data that are required for supervisory purposes. They include

Bank	Market Share	Cumulative Share
Bank Austria Creditanstalt	16.75%	16.75%
Erste Bank AG	10.19%	26.94%
RZB	7.89%	34.83%
BAWAG PSK	7.21%	42.04%
Kontrollbank	3.64%	45.68%
OeVAG	2.20%	47.88%
RLB OOE	2.15%	50.03%
Kommunalkredit	2.13%	52.16%
Hypo Alpe Adria	2.06%	54.23%
RLB NOE Wien	1.95%	56.18%

Table 1.1. *The 10 largest Austrian banks in 2005 and their respective market shares in terms of total assets.*

numbers on capital adequacy statistics, interest rate sensitivity of loans and deposits with respect to various maturity buckets and currencies, and foreign exchange exposures with respect to different currencies.

In our analysis we use a cross section from the MAUS database of all reporting banks in the relevant observation period.

In December 2005 the aggregate total assets of the Austrian Banks amount to 725 billion Euro. This is approximately 2.7 times the Austrian GDP in 2005. The banking industry is highly concentrated. As we see in Table 1.1 the two largest banks account for a quarter of aggregate total assets. The market share of the 10 largest banks is more than 55%. The domestic interbank liabilities (deposits) amount to 113 billion Euro (120 billion Euro). The liabilities against foreign banks sum up to 98 billion Euro (deposits 88 billion Euro). The share of interbank exposures to total assets varies a lot across banks from 0% up to 96%. The average (median) exposure is 13% (8%).

To estimate shocks on bank capital stemming from market risk, we include positions in foreign currency, equity, and interest rate sensitive instruments from MAUS. For each bank, we collect foreign exchange exposures for USD, JPY, GBP, and CHF only, as no bank in our sample has open positions of more than 1% of total assets in any other currency at the observation date. We collect exposures to foreign and domestic stocks, which are equal to the market value of the net position held in these categories. For the exposure to interest rate risk we use the interest rate risk statistics, which provides exposures of all interest sensitive on- and off balance sheet assets and liabilities with respect to 13 maturity buckets for EUR, USD, JPY, GBP, CHF, and a residual representing all other currencies. On the basis of this information we calculate net positions in the available currencies - neglecting the residual - with respect to four different maturity buckets: up to 6 months, 6 months to 3 years, 3 to 7 years, more than 7 years. For the valuation of net positions in these maturity buckets we use the 3 month, 1 year, 5 years and 10 years interest rates in the respective currencies.

This procedure gives us a vector of 26 exposures, 4 FX, 2 equity, and 20 interest rates (4 maturities for each currency), for each bank. Thus we get a $N \times 26$ matrix of market risk exposure.

To analyze credit risk we use in addition to the data provided by MAUS the major loans register of OeNB (GKE) which provides us with detailed information on the banks' loan portfolios to non-banks. This database contains all loans exceeding a volume of 350, 000 Euro on a loan by loan basis.[1]

We assign the domestic loans to non-banks to 13 industry sectors (basic industries, production, energy, construction, trading, tourism, transport, financial services, public services, other services, health, private households, and a residual sector) based on the NACE-classification of the debtors. Furthermore we add regional sectors (Western Europe, Central and Eastern Europe, North America, Latin America and Carribean, Mid East, Asia and Far East, Pacific, Africa, and a residual sector) for foreign banks and non-banks individually, resulting in a total of 18 non-domestic sectors. Since only loans above a threshold volume are reported to the GKE we assign domestic loans above this threshold to the domestic residual sector. This is done on the basis of a report that is part of MAUS and provides the number of loans to domestic non-banks with respect to different volume buckets. For non-domestic loans no comparable statistic is available. However, one can assume that most of cross-border lending exceeds the threshold of 350, 000 Euro and hence the associated risk can be neglected.

The riskiness of an individual loan to domestic customers is assumed to be characterized by two components: the rating which is assigned by the bank to the respective customer and the default frequency of the industry sector the customer belongs to. The banks rating is reported to the GKE and is mapped within the OeNB on a master scale, which allows assigning a probability of default to each loan. The default frequency data are from the Austrian rating agency Kreditschutzverband (KSV). The KSV database provides us with time series of insolvencies and the total number of firms in most NACE branches at a quarterly frequency starting in 1969. This allows us to calculate a time series of historically observed default frequencies for our 13 industry sectors by dividing the number of insolvencies by the number of total firms for each industry sector and quarter. The time series of default frequencies is explained by macroeconomic risk factor changes using an econometric model. By this estimated equation we can translate macroeconomic risk factor changes in probabilities of default for each industry branch. These default probabilities serve as input to the credit risk model. To construct insolvency statistics for the private and the residual sector, where no reliable information on number of insolvencies and sample size is available, we take averages from the data that are available. Default probabilities for the non-domestic sectors are calculated as averages of the default probabilities according to the ratings that are assigned by all banks to all customers within a given foreign sector.

1.4 Applications

OeNB uses the SRM model mainly for two applications: Systemic risk assessment and stress testing. Systemic risk assessment performs a simulation at the beginning of each quarter as soon as all new data are available. The output of this simulation is a risk report containing system wide measures of default probabilities of Austrian banks and a quantification of potential shortfalls that might have to be covered if wide spread default occurs. In the Stress tests one or more risk factors of interest are deliberately set to an extreme value and the simulation is performed conditional on the assumption that these risk factors are at their hypothetical extreme realizations. The output of this simulation can then be compared to the baseline simulation.

[1]The GKE database covers about two third of all loans of Austrian banks in terms of nominal values.

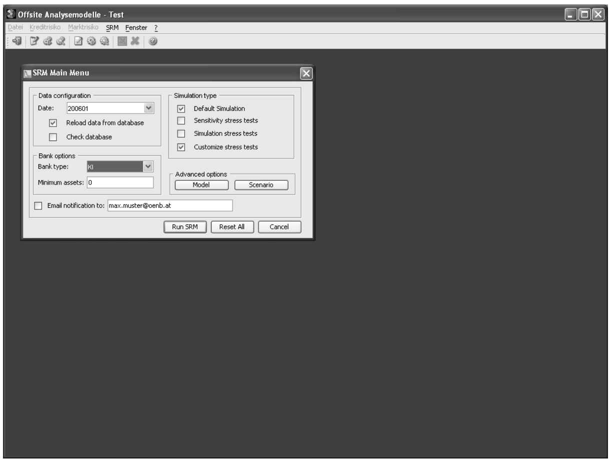

Figure 1.2. *A screenshot from the SRM interface.*

To make SRM operational, it is implemented such that it can be accessed via an interface that can be called from the analyst's desk. The interface is a Java client application which gives users the possibility to run certain predefined simulations (including a variety of regular stress tests) as well as to parameterize individual simulations. The level of parameterization covers the point in time for which the simulation is run, data included in the model, various alternative model components, as well as their parameters. Additionally stress tests can be defined for market and credit risk factors. Parameters chosen are stored at database level and written to configuration files, which are read by the application at runtime. The models themselves are implemented with Matlab, version 14.3, a programming language for technical computing, which provides object-oriented means to include various model components and store complex data sets. Although SRM functionality can be accessed through Matlabs standard user interface, in its end-user implementation the source code of SRM is compiled as C Code and called via the SRM interface. In either case output is written to Microsoft Excel files for further analysis, which are sent as an e-mail attachment to the analysts desk by SRM after a simulation request has been finished. A screen shot from the interface is shown in figure 1.2.

1.4.1 Regular Supervisory Data Analysis and Stress Tests

Systemic Risk Monitor will be used to do regular analysis of supervisory data with respect to systemic risk problems. It will also be used as a stress testing tool. There is no coherent approach to stress testing so far. While most stress testing exercises in the past were focused on the analysis of single stress scenarios, SRM allows to do stress tests via full simulations that take the dependence between risk factor changes explicitly into account. Instead of analyzing a single stress scenario, we can set one or more risk factors to their hypothetical stressed value and simulate from the conditional distribution of risk factor changes. This approach has been advocated by Elsinger et al. [2006b]. It has also been proposed by risk modelers from the banking industry (see for example Bonti et al. [2005]).

Of course this approach to stress testing does not exclude more traditional single scenario stress tests, that have frequently been used in the past in particular to support the IMF's Financial Sector Assessment Program (FSAP). The modular structure of SRM allows to side step the risk factor distribution model and perform a network clearing under the assumption that one risk factor or some other parameter of interest is at its stressed value while all other risk factor values and parameters are at their default values. SRM is therefore able to nest traditional stress testing techniques in its overall framework.

We will now illustrate output generated by SRM by looking at some examples based on a recent simulation for the last quarter of 2005. We present our results always for a regular simulation, based on the date of the last quarter 2005 and two stress tests: Stress test number one simulates an unexpected drop in GDP. Stress test number two assumes a rise in the three month Euro interest rate by 120 basis points.

1.4.2 Fundamental and Contagious Defaults

The network model generates a multivariate distribution of bank insolvencies across scenarios. This multivariate distribution contains information on the marginal distributions of individual bank defaults as well as on default dependency among the banks. We interpret the relative frequency of default across scenarios as a default probability.

Our method allows a decomposition of bank insolvency cases into those resulting directly from shocks to the risk factors and those that are consequences of a domino effect. Bank defaults may be driven by losses from market and credit risk, (fundamental default). Bank defaults may, however, also be initiated by contagion: as a consequence of other bank failures in the system (contagious default).

We can quantify these different cases and are able to give a decomposition into fundamental and contagious defaults. Table 1.4.2 summarizes the probabilities of fundamental and contagious defaults both in the basic simulation as well as under both stress scenarios. These probabilities are grouped by the number of fundamentally defaulting banks.

Tabel 1.4.2 shows that in the base case simulation we have no scenario where in total more than 5 banks will default fundamentally. Among all the scenarios including up to 5 fundamental defaults all scenarios show no contagion. This is result is consistent with the findings in Elsinger et al. [2006a] where it is shown that contagion is a rare event given a risk factor change distribution calibrated to historical data. In situations of stress the picture changes. When we have a drop in GDP where up to 50 banks default fundamentally and there can also be some contagion once we have 21 to 50 fundamental defaults. The stress test for an interest rate hike looks less spectacular. The simulation shows no contagion effects but at least one and up to at most five banks are expected to default. Thus according to the simulation we have to expect at least one default with certainty under such a stress scenario. The analyst using SRM has the opportunity to look deeper into the micro structure of these results and find out details about the institutions that are most severely hit under the stress scenario.

1.4.3 Distribution of PD according to Rating Classes

Table 1.4.2 gives us the aggregate picture. To get a more precise picture about the distribution of risk within the banking system we map the probabilities of default

Number of Banks	Base Case fund.	Base Case cont.	GDP-Stress fund.	GDP-Stress cont.	Interest-Stress fund.	Interest-Stress cont.
0	74.49%	0.00 %	68.53%	0.00%	0.00	0.00 %
1 to 5	25.51%	0.00 %	31.27%	0.00%	100.00	0.00 %
6 to 10	0.00%	0.00 %	0.13%	0.00%	0.00	0.00 %
11 to 20	0.00%	0.00 %	0.05%	0.00%	0.00	0.00 %
21 to 50	0.00%	0.00 %	0.02%	0.02%	0.00	0.00 %
more than 50	0.00%	0.00 %	0.00%	0.00%	0.00	0.00 %
total	100.00%	0.00 %	100.00%	0.02%	100.00	0.00 %

Table 1.2. *Probabilities of fundamental and contagious defaults. A fundamental default is due to the losses arising from exposures to market risk and non-bank credit risk, while a contagious default is triggered by the default of another bank that cannot fulfill its promises in the inter-bank market. The probability of occurrence of fundamental defaults alone and concurrently with contagious defaults is observed. The observation period is December 2005. The time horizon is one quarter. The column Base Case shows the result for a simulation without stress. The Column GDP-Stress shows the case of a stress test with an unexpected drop in GDP. The third column Interest-Stress shows the stress test with a 120 basis point increase in the short term (three month) Euro interest rate.*

into the rating classes used by OeNB, which has seven non-default rating-classes and eight default classes. This distribution of ratings that is implied by our simulation can be seen in Table 1.4.3.

Class	OeNB MS	Base Case abs.	Base Case rel.	GDP-Stress abs.	GDP-Stress rel.	Interest-Stress abs.	Interest-Stress rel.
1	AAA	800	94.67%	779	92.19%	791	93.61 %
2	AA	0	0.00%	0	0.00%	0	0.00 %
3	A	8	0.95%	13	1.54%	7	0.83 %
4	BBB	15	1.78%	22	2.60%	15	1.78 %
5	BB	13	1.54%	19	2.25%	15	1.78 %
6	B	8	0.95%	9	1.07%	14	1.66 %
7	C	1	0.12%	3	0.36%	2	0.24 %

Table 1.3. *Share of banks in OeNB rating classes.*

Table 1.4.3 shows that in the base case simulation about 95% of banks are expected to be in a triple A rating at the end of the first quarter of 2006. Under the assumptions about our two stress scenarios the number of top rated institutions decreases slightly. The biggest increase under stress can be observed in the lower rating classes. The number of banks and the rating class just above the default class triples in the first stress scenario (drop in GDP) and doubles in the second (increase in the Euro interest rate).

1.4.4 Aggregate Loss Distributions

Going from insolvencies to the distribution of losses over the next quarter we can draw pictures of the losses due to credit and market risk as well as due to the combination of both losses. Contrary to familiar pictures from the practice of risk management these distributions are derived from an integrated analysis of all portfolio positions and its change in value due to the entire distribution of risk factor changes. Thus rather than analyzing credit and market risk in isolation these graphs give us the results from an integrated analysis. Figure 1.4.4 shows four loss distributions. From the figures we can see – as in standard quantitative risk management – whether or not the system has enough capital to absorb extreme losses. Therefore loss distribution figures give a first overview of the shock absorption capacity of the system.

1.4.5 Value at Risk for the Lender of Last Resort

A relevant aspect of our model for the regulator is that it can be used to estimate the cost of crisis intervention. We estimate the funds that would have to be available to avoid contagious defaults or even fundamental defaults for different confidence levels. A lender of last resort's cost of preventing fundamental default is calculated as the amount required to prevent banks from becoming insolvent. A lender of last resort's cost of preventing contagious defaults is calculated as the amount required to prevent all but fundamentally defaulting banks from becoming insolvent. Hence, interbank liabilities are not fully insured but just enough to prevent contagion. Table 1.4 reports our results for the base line simulation

	Base Case		GDP-Stress		Interest-Stress	
Quantiles	95%	99.5%	95%	99.5 %	95%	99.5%
Resources	29.16	33.16	29.16	101.34	1.24	29.76

Table 1.4. *Costs of avoiding defaults: In the first row we give estimates for the* $95, 99$*, and* 99.5 *percentile of the avoidance cost distribution across scenarios. Amounts are in million euros.*

Since defaults occur rarely in the base scenario the amounts that must be available to prevent default in most of the scenarios are low. In a stress the amount of funds that have to be mobilized by a lender of last resort increase but they remain still very low. The analysis shows that for the particular quarter of December 2005 a lender of last resort can expect that even if crises scenarios simulated by the model do actually occur, in case of crises intervention the amounts to be mobilized will be small.

1.4.6 Changes in System wide VaR under Stress

Finally we analyze the changes in Value at Risk of the distribution of losses relative to regulatory capital. That is we look at the distribution of losses in percent of regulatory capital and look at the quantiles of this distribution. In our case we analyze the 99% quantile or the 99% value at risk. We look at these measures for the different subcategories, total losses, market losses, credit losses and contagion losses. The results are reported in Table 1.5.

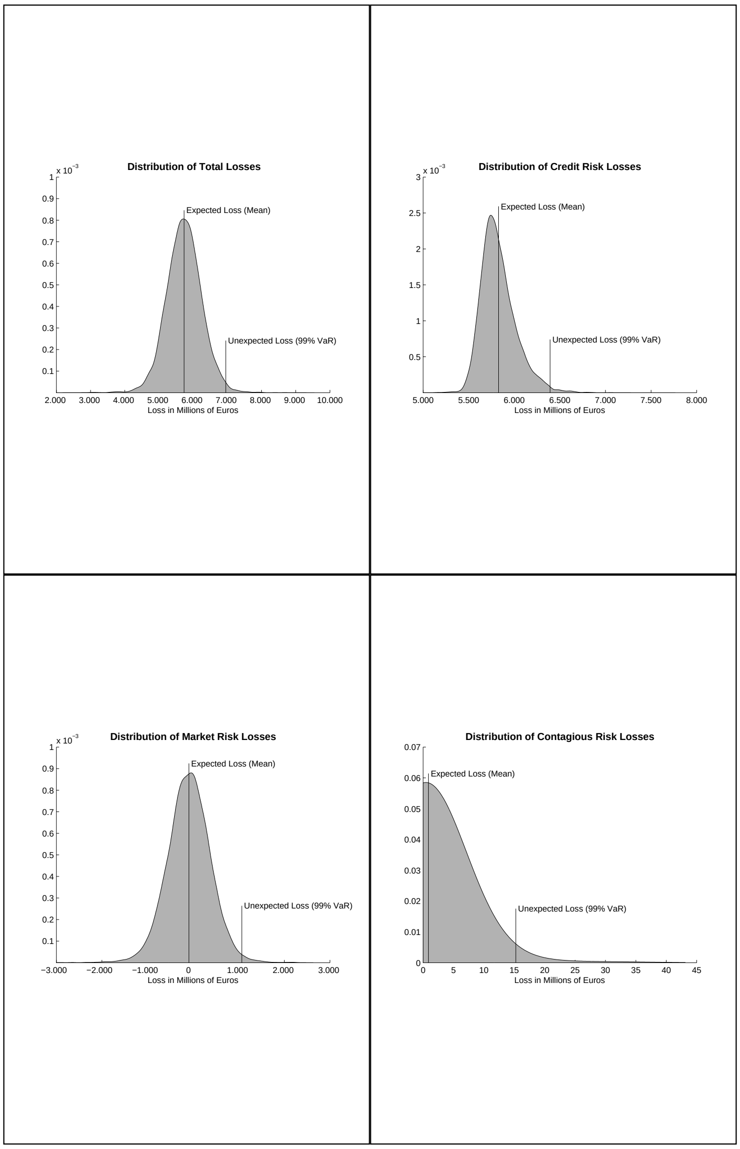

Figure 1.3. *Densities of the loss distribution for the whole banking system. The densities are shown for the entire portfolio and separated according to market and credit risk as well as according to the losses due to contagion.*

Rel. VaR	Total	Market	Credit	Contagion
Base Case	13.58 %	2.11 %	12.35 %	0.03 %
GDP-Stress	16.95 %	5.68 %	12.52 %	0.05 %
Interest-Stress	15.56 %	4.34 %	12.40 %	0.04 %

Table 1.5. 99% *quantile of the distribution of losses relative to regulatory capital for Total losses, losses from market risk, losses from credit risk and losses from contagion risk. This relative VaR is shown for the baseline simulation, fro the case of a GDP-stress test and for the case of an interest rate stress test.*

Chapter 2

The Multivariate Distribution of Risk Factor Changes

In SRM uncertainty is described by a combination of exogenous and endogenous risk. The exogenous risk drivers are captured by a multivariate distribution of risk factor changes that affect the value of the non-interbank positions. The value of interbank exposures is determined endogenously by the network model via a clearing procedure for all financial claims in the system.

The choice of risk factors is determined by the data that are available to describe bank portfolios. On the one hand we have aggregate positions of marketable securities such as stocks, bonds, deposits and securitized liabilities. For all these categories we have both domestic and foreign positions. The value of these exposures depends on the change of market prices: stock market indices, interest rates at different maturities and for different currencies as well as exchange rates. We choose risk factors in these categories that can be mapped to the exposure information that we can gain from our data. Loans, which constitute a major part of the bank assets are not traded on markets and can not simply be valued at market prices. Risk factors that drive the value for loans are macroeconomic fundamentals that influence corporate default rates and thus the quality of the loan portfolios of banks.

The joint distribution of all relevant risk factors, market prices and macroeconomic variables that drive corporate defaults, describes the exogenous risks and their impact on banks simultaneously. In SRM the distribution is needed for simulation. It should enable us to draw as many risk scenarios as we need to get a good Monte Carlo simulation for the distribution of fundamental and contagious defaults as well as of the distribution of market and credit risk losses in the system. Modeling such a distribution poses several challenges.

The first challenge is that the integrated analysis of market and credit risk factors requires a common time horizon . It is not straightforward which common horizon is adequate because market positions can be turned over at very short frequencies whereas positions in the loan book can usually not be turned over quickly and are held for longer horizons. In SRM we use a common time horizon of both market and credit exposures of one quarter which is approximately 60 trading days. This choice of time horizon leaves us with the problem how to integrate market data, that are available at a daily frequency, with data related to the risk of credit exposures, which are only available quarterly.

The second challenge is the adequate description of marginal distributions of risk factor changes. Since we want to use the multivariate distribution model for simulating portfolio gains and losses over a three month horizon we choose the models for the marginal risk factor change distributions based on their out of sample performance rather than their in sample fit. The ultimate purpose of SRM to simulate fundamental and contagious defaults requires that we forecast the whole distribution and not only some of its moments. Our criterion is thus to choose from a class of a priori reasonable models one that gives us the best out of sample performance in forecasting the multivariate distribution of risk factor changes over a three month horizon.

The third challenge is to capture dependence in an adequate way. Since we saw that stylized facts about financial market data often show deviations from normality, we need a model of dependence that is able to capture this fact. Working with a framework that is both flexible and tractable we decided to work with a grouped t-copula for dependence modeling.

2.1 Risk factors: Stylized facts

Risk factors in SRM consist of logarithmic daily changes (returns) of financial market prices and of quarterly macroeconomic variables. In the model both set of variables are brought to a common time horizon of a quarter. Most of the risk factors are time series data of daily market returns. Aggregation to lower frequencies, such as a quarter are based on these series.

To better see the empirical motivation for our approach to modelling the multivariate distribution of changes in risk factors it might be helpful to consider some stylized facts about financial time series. In the following discussion we confine ourselves to an illustration of these facts to one example from our data set that shows these stylized facts most clearly rather than giving a systematic discussion of all the risk factors used. These stylized facts are well known in empirical finance and are summarized by McNeil et al. [2005] in chapter 4 of their book.

The first fact is that return series show varying volatility over time and are not iid (identically independently distributed) although they show little serial correlation. Figure 2.1 shows *daily* log returns of the DAX from 1990 to 2004 and simulated iid data from a normal distribution calibrated to the parameters of the return data. The simulated normal data are visibly different from the empirical data.

Compared to the simulated series the real series shows the tendency of extreme returns followed closely by other extreme returns, a phenomenon known as *volatility clustering*. If we look at correlograms of the raw data and their absolute values as well as on the correlogram of the raw data and the simulated data in figure 2.2 we see very little autocorrelation in the raw data but contrary to the simulated data the raw data show autocorrelation in the absolute values, a sign that the data are not independent.

These features can also be found in foreign exchange as well as in all interest rate series, we use in SRM. It is interesting that the correlogram shows little or no autocorrelation in the raw data. This expresses a further stylized fact, namely that conditional expected returns are close to zero. The presence of volatility clustering suggests however that conditional standard deviations are continuously changing over time. Thus to capture the nature of financial return series we often need to model changing volatility.

The final set of stylized facts relates to extreme values. 2.3 shows the empir-

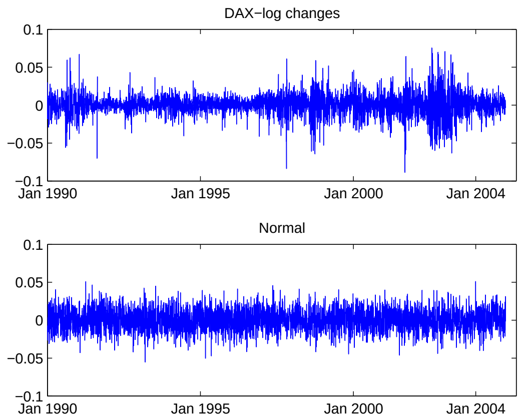

Figure 2.1. *Log returns from DAX index from January 2nd 1990 to December 31 2004 compared with simulated iid data from a normal distribution.*

ical quantiles of Dax returns against the quantiles of a normal distribution. The deviations from the dashed line shows that the data tend to have heavier tails than the normal distribution.

These facts apply to other financial time series as well but to different degrees. Furthermore as we decrease the frequency to lower time intervals the features shown here with respect to the DAX data become less pronounced.

Our approach to the multivariate distribution model of risk factor changes is motivated by the aim to accommodate these features of the data if necessary, and to allow a certain discrimination or grouping between different categories of risk factors, such that we can differentiate whether we describe stocks, interest rates, foreign exchange or macro variables.

2.2 A Copula Approach

In SRM the realizations of many different risk factor changes determine the portfolio values and thus the solvency or insolvency of banks in different scenarios. Systemically important events – risk factor changes that have a potentially high impact on the banking system – are intimately related to the *dependence* between these risk factor changes. It is for instance a stylized fact about banking crises in the past that episodes of joint bank defaults occur together with high loan default rates and a situation of depressed asset values (Borio [2003], Goodhart et al. [2003]) In such a situation risk factor changes that have an impact on credit risk and risk factor changes that have an impact on market risk move simultaneously in a direction that bring many institutions under pressure at the same time.

At an abstract level the dependence between changes in risk factors $X_1, \ldots, X_n$ is completely described by their joint distribution

$$F(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

If the number of risk factors is large, modeling this distribution is a challenging task. One is typically in a situation where relatively few observations are available to

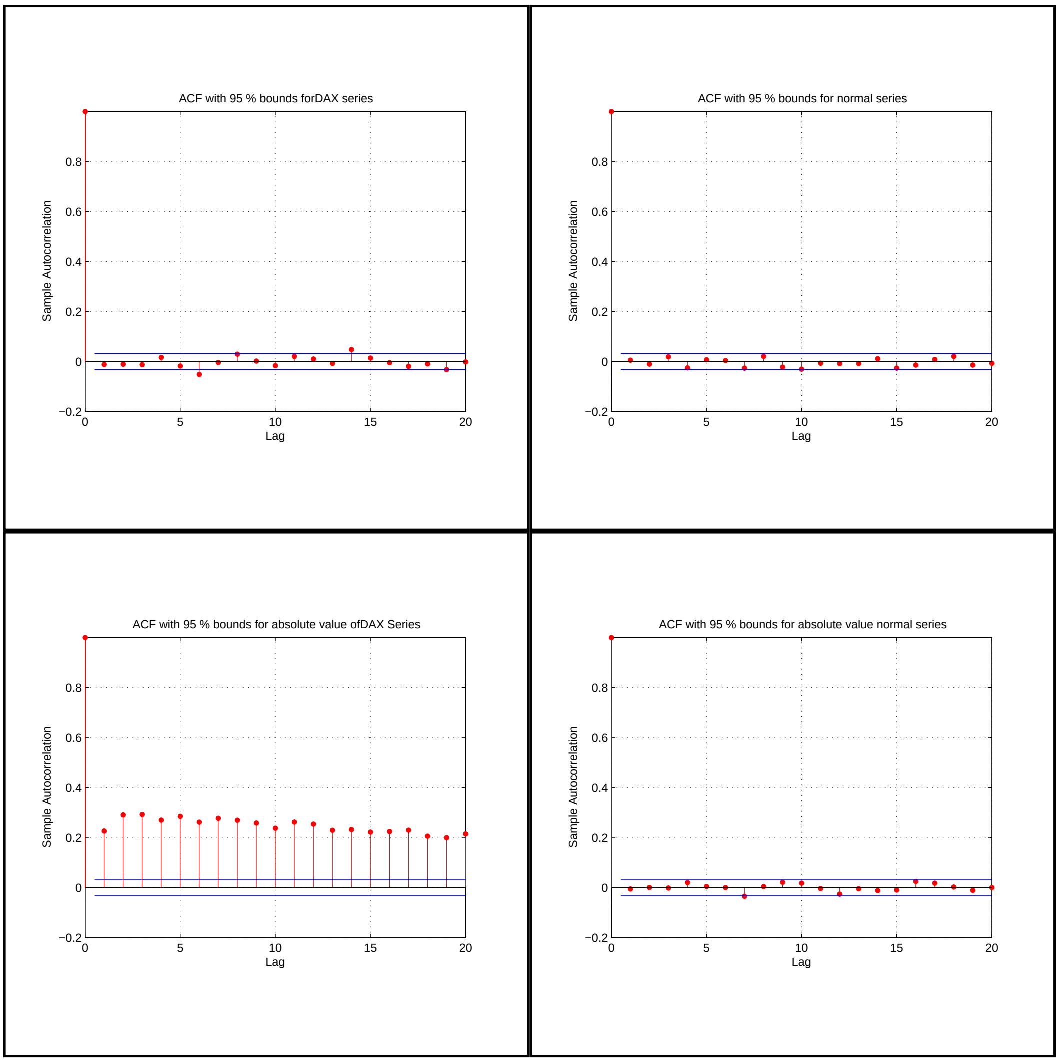

Figure 2.2. *Correlograms for the two data sets. Dashed lines mark the standard 95 % confidence interval for autocorrelations of a process of iid finite variance random variables.*

fit a very high dimensional problem. For instance, if one works with the assumption that the risk factor changes are best described by a 20 dimensional multivariate normal distribution and we have only quarterly historical data from 1970 until 2005, we have 210 parameters that are estimated with only 2800 observations. The data situation can be improved as we collect more observations. As soon as we want to capture features in the data that deviate from the normal distribution assumption we need distribution models that require to fit additional parameters. Since market return data, in particular at shorter frequencies (like daily or weekly intervals) deviate substantially from normality a reasonable model must be able to accommodate these features in the data.

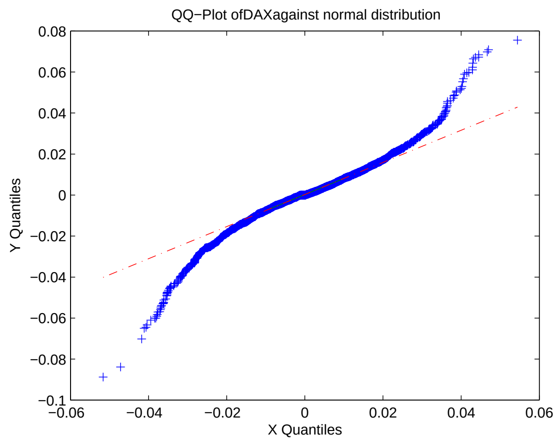

Figure 2.3. *The figure shows a plot of the empirical quantiles of the DAX against the quantiles from data simulated from a normal distribution. The plot shows an inverted S-shape, which is evidence of substantial deviations from normality in the tails of the distribution.*

The copula approach allows a "divide and conquer" strategy to model the multivariate distribution and allows at the same time to accommodate potential deviations from normality that might exist in our data.

Before we explain the details of this approach and how we apply it to our problem, it might be helpful to explain the divide and conquer strategy in the familiar context of a normal distribution. Instead of estimating all the parameters simultaneously we could split the problem in two parts, where we first would estimate the means and the variances of all risk factors separately and then estimate the correlation coefficients. In other words, we first model all the marginal distributions and then consider their dependence separately. For more complex dependency structures, as we usually have them in problems like SRM, this approach does not help (see Embrechts et al. [2002]). We need a more general model of dependence.

The approach we choose in SRM rests on a generalization of the two stage strategy used in the case of the normal distribution. We first look only at models for the univariate margins of risk factor changes. The issue of dependence is then studied separately by using the concept of a copula. The example of the multivariate normal distribution is special because only in this case the dependence structure is characterized by the correlation. If we want to study dependence in a broader context, permitting also risk factor change distributions which deviate from normality, the copula approach turns out to be very useful.

The approach works such that we first capture the marginal distributions by different statistical models $F_1(x_1), \ldots, F_n(x_n)$ for each risk factor separately. This splits the task in the first step into n one dimensional problems for each statistical model for the marginals. For each of these models for the univariate margins F_i we then select the one that gives the best *out of sample* density forecast over a three month horizon.

Once the models for the marginal distributions are chosen, the question of dependence between the single risk factors is modelled separately by a copula function.

The marginal distributions and the copula in combination then uniquely determine the multivariate distribution $F(x_1, \dots, x_n)$, a fact that is known as Sklar's theorem (see Nelsen [1999]).

Let us start with the more familiar problem of modelling the marginal distributions of risk factor changes and then explain the more advanced concept of modeling dependence by a copula in a second step.

2.2.1 Modeling Marginal Distributions

When choosing a model for the marginal distributions we try to work with as few a priori assumptions about distributional properties as possible. Instead we rely on statistical out of sample tests of the 60 day density forecasts produced by a group of reasonable models.

Based on these test results we select a small set of models able to cover all the relevant time series, rather than picking for each time series the optimal model. The motivation for this strategy is parsimony and robustness under the inclusion of new time series. We want to avoid that all models have to be tested on a new time series whenever a new time series is included.

The risk factors in SRM are logarithmic changes in market prices on the one hand and in a set of macroeconomic variables on the other hand. For market prices we look directly at returns. For some market price models we work with models of changing volatility and tale the residuals of these models as an input for the multivariate distribution of risk factor changes. For the macrofactors we look at residuals of a Vector Autoregression of log-changes of macroeconomic variables that have an impact of credit risk for corporate loans. While financial return data show very little autocorrelation, for macroeconomic risk factor changes there is usually a significant impact of lagged values of all macroeconomic risk factor changes on contemporaneous values and this dependence has to be taken into account in the modeling strategy. We have first to estimate these systematic dependencies and take the residuals of these estimates as an input for the copula model of dependence. Thus we are looking for models of logarithmic changes of market prices on the one hand and of the residuals of a Vector Auto Regression of logarithmic changes of a set of macroeconomic variables on the other hand.

Concerning the time horizon we want to consider all risk factor changes at a quarterly (three months) time horizon. Since market data are available at a higher frequency we consider the possibility of aggregating the information contained in higher frequency market data to the lower frequency quarterly time horizon.

Specifically we consider the possibility to model changes of market prices over shorter periods of 1, 5, 10, 20, or 30 days, and then aggregate these changes over 60, 12, 6, 3, or 2 periods. In this way we can exploit the availability of higher frequency data in order to get more reliable estimates. On the other hand, estimation errors will be magnified by the aggregation procedure. Not knowing which effect is stronger we have tested the 60 day density forecasts produced by aggregating estimated distributions for various basic periods.

We use the following models for the marginal distributions of market risk factor changes and for the residuals of the vector auto regression of the macroeconomic risk factors. The first model uses the assumption that the risk factor changes come from a normal distribution. The second model assumes that these changes have a t-distribution. Finally we assume that the risk factor changes come from a model where the body of the distribution is estimated by a kernel estimator and the lowest 10 % as well as the highest 10 % of changes are estimated by an extreme value estimator based on the generalized Pareto distribution. This distributional assumption is motivated by limit results in extreme value theory and is frequently used in risk management applications. This procedure is for instance used in McNeil and Frey [2000].

To allow for the consideration of stochastic volatility and the clustering of extreme movements in risk factor changes – as it is often observed in financial market returns data – we also consider univariate GARCH(1,1) models.

Combining all these features we get a set of models for the market risk factors and the residuals from the macroeconomic Vector Auto Regression. The models are shown in Table 2.1. The terminology of the model names listed in Table 2.1 should be read as follows:

- xxd(m) means that the model aggregates distributions for xx day (months) changes to arrive at the 60 day distribution.
- G in the model name means that the GARCH(1,1) is used.
- The last part of the model name represents the distributions of returns or errors: t for the Student, norm for the normal distribution, and EVT for the kernel plus extreme value distribution.

In this terminology all the models for the marginals of market risk factors are summarized in Table 2.1.

1d_norm	1d_t	1d_EVT	1d_G_norm	1d_G_t	1d_G_EVT
5d_norm	5d_t	5d_EVT	5d_G_norm	5d_G_t	5d_G_EVT
10d_norm	10d_t	10d_EVT	10d_G_norm	10d_G_t	10d_G_EVT
20d_norm	20d_t	20d_EVT	20d_G_norm	20d_G_t	20d_G_EVT
30d_norm	30d_t	30d_EVT	30d_G_norm	30d_G_t	30d_G_EVT
60d_norm	60d_t	60d_EVT	60d_G_norm	60d_G_t	60d_G_EVT

Table 2.1. *All models considered for the density forecast of market risk factor changes.*

Since macroeconomic data are available only at a quarterly basis, aggregation of higher frequency distributions is not an option. So the models for the marginals of the residuals of the VAR model for macroeconomic risk factors are:

3m_norm	3m_t	3m_EVT	3m_G_norm	3m_G_t	3m_G_EVT

Table 2.2. *All models considered for the density forecast of macroeconomic risk factor changes.*

2.2.2 Testing Density Forecasts for Different Models of Marginal Risk Factor Change Distributions

Scenarios in SRM are produced by simulation from the multivariate distribution of risk factor changes. In order to do so we need to have the right overall distributional properties, not just the right means or covariances. Therefore we have to test statistically for the adequacy of the *distributions forecasts* for all models of the marginals as described in Tables 2.1 and 2.2. Our test procedure is based on the work of DeRaaij and Raunig [2002].

The technical details of the tests are described in the appendix. The general idea of our testing procedure is based on the theory of density forecast tests based on the so called probability integral transform (see Rosenblatt [1952]). The idea of this distributional test is based on a simple fact from probability theory: Given a random variable X with distribution function F, the random variable $U = F(x)$ is uniformly distributed on the interval $[0, 1]$. By definition of a distribution function, F can only take values between 0 and 1. The fact that the values of F have to be uniformly distributed also follows directly from the definition of a distribution function. If $0 \leq u \leq 1$, we have

$$\Pr(F(x) \leq u) = \Pr(x \leq F^{-1}(u)) = F(F^{-1}(u)) = u$$

This argument is rigorous when the distribution function F is strictly increasing. Thus if our model of the distribution F_M coincides with the true distribution F_T of the data the distribution of the probability transformed data must be uniform. For the construction of statistical tests it is often convenient to transform the uniform data to the normal distribution by using a quantile transform suggested by Berkowitz [2001]. Using this (additional) transformation, the transformed variables must be distributed according to a standard normal distribution if the model distribution matches the true distribution.

The tests we use for model selection are based on this idea. We use two tests to check whether the transformed data under the different models of the marginal distributions satisfy normality. One is the Kolmogorov-Smirnov test. This test compares the sample values with the values of a standard normal distribution. The Null Hypothesis of the test is that the sample values follow a standard normal distribution. The alternative hypothesis is that the sample does not have a standard normal distribution. Intuitively the Kolmogorov Smirnow test is based on considering the whole distribution. Its disadvantage is that it is not very powerful against deviations from the Null Hypothesis. Therefore we use additionally an alternative route which is based on work of DeRaaij and Raunig [2002]. Their procedure is more focused on moments rather than on the entire distribution. We rank our models for the marginals according to the criterion for how many risk factors jointly pass the Kolmogorov Smirnov test and the tests of DeRaaij and Raunig [2002]. The models with the highest scores in this ranking are selected. For a technical discussion of the test procedures we refer the reader to the appendix. Based on this test procedure the 3m_G_EVT model gave overall the best out of sample distribution forecasts and is therefore used in SRM. In the practical use of SRM we do not make a model selection in every new quarter based on this test procedure. We do however update the model selection in regular longer intervals.

2.2.3 The Grouped t-Copula

To explain the idea of a copula we recall the fact that for a random variable X it must be true that $F(x)$ is uniformly distributed on $[0, 1]$. Using this fact we can use the marginal distribution function of each risk factor to transform it to a uniformly distributed random variable on the interval $[0, 1]$, by evaluating each F_i at the sample points. This transformation removes the influences of the original marginal distributions and leaves only the intrinsic dependence between variables.

This discussion motivates the abstract definition of a copula: A copula is the joint distribution of uniformly distributed random variables. If $U_1, \ldots, U_n$ are $U(0,1)$, then the function C from $\times_{i=1}^{n}[0,1] \to [0,1]$ defined by

$$C(u_1, \ldots, u_n) = Pr(U_1 \leq u_1, \ldots, U_n \leq u_n)$$

is a copula. Moreover, if $X_1, \ldots, X_n$ are random variables with distribution functions $F_1, \ldots, F_n$ respectively, then the copula of the uniform random variables

$$U_1 = F_1(X_1), \ldots, U_n = F_n(X_n) \tag{2.1}$$

is called the copula of $(X_1, \ldots, X_n)$.

An important example of a copula, which we build on in SRM is the d-dimensional t-copula which takes the form

$$C^t_{\nu,P}(u) = t_{\nu,P}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n)\right) \tag{2.2}$$

where t_ν is the distribution function of a standard univariate t distribution and $t_{\nu,P}$ is the joint distribution function of the vector $X \sim t_d(\nu, 0, P)$ and P is a correlation matrix.

The *simulation* of a t-copula is relatively easy. First we have to simulate a random vector $X \sim t_d(\nu, 0, P)$. Then return a vector $U = (t_\nu(X_1), ..., t_\nu(X_d))$ where t_ν denotes the distribution function of a standard univariate t distribution. The random vector U has distribution function $C^t_{t,P}$. Details on the simulation of multivariate t-distributed random vectors can be found in Demarta and McNeil [2004] or in McNeil et al. [2005].

The grouped t-copula has been introduced by Daul et al. [2004]. It is a copula closely related to the t-copula where different sub-vectors of the vector X of risk factor changes can have different levels of tail dependence. In the grouped t-copula the marginal distributions $F_1(X_1), \ldots, F_n(X_n)$ have univariate t distributions but with different degree of freedom parameters $\nu_1, \ldots, \nu_n$. The rationale is that risk factors can be grouped by their dependence properties described by the same ν_j parameter, while using established techniques known from the estimation and calibration of t-copulas.

Such an approach has a natural application to SRM where we have different categories or risk factors: changes in stock indices, changes in foreign exchange and changes in interest rates of different maturities and different currencies as well as the residuals from a vector auto regression of macroeconomic variables.

Logarithmic changes in market prices are known to usually show deviations from normality, in particular heavy tails which can be captured by the grouped t-copula. If some data groups are close to normality this feature is also picked up by the grouped t-copula because the normal distribution is the limit of a t-distribution for the degrees of freedom going to infinity. For practical purposes this means degrees of freedom of larger than 30.

The grouped t-copula is a parametric copula and can be relatively easily fitted to data, once the models for the marginal distributions are specified. The exact procedure how the fitting is done is explained in detail in the technical appendix.

We considered two possible groupings. Grouping A is: (A1) all macro time series which were used in the VAR model, (A2) all market time series. Grouping B is: (B1) all macro time series which were used in the VAR model, (B2) all interest rate time series, (B3) all FX time series, (B4) all equity time series. In our implementation we use grouping (B).

For reasons of computational efficiency but also for reasons of stress testing it is desirable if we are able to simulate scenarios not only from the unconditional copula but also from the conditional multivariate distribution of risk factor changes of risk factor changes or the conditional copula. How such conditioning can be built into the procedures is also described in the technical appendix.

Chapter 3

The Market Risk Model

Following standard methods from risk management (see Jorion [2000]), we construct a mapping from market risk factors to portfolio positions based on the MAUS database.

Contrary to related problems in the banking industry where market risk is modelled at a fairly short time horizon with a very detailed picture of individual instruments and positions, our portfolios that can be reconstructed from MAUS is much coarser and considered over a time horizon of a quarter (approximately 60 trading days).

Our basic idea is to take all portfolio positions we can reconstruct from MAUS and that we can relate to an appropriate market risk factor and simulate the gains and losses to these portfolios as a part of the overall gains and losses of the banks in the system.

A detailed description of the particular risk factors used in SRM is given in chapter 6. We identify from MAUS positions of stocks in domestic and foreign equity. These positions are mapped to a domestic stock market index and a world stock market index. We can also identify broad maturity buckets of three months, one, five and ten years of interest rate sensitive exposures as well as exposures denominated in the five currencies most important for Austrian banks, USD, JPY, GBP and CHF. For all interest rate sensitive foreign exposures we include the appropriate term structure of interest rate according to the maturity buckets we can construct from the MAUS database.

The change in value of instruments sensitive to market risk factor changes is determined by a first order approximation of the market portfolio loss. To explain this concept, remember the notation for portfolio losses we have developed in the introduction. Denoting the vector of market risk factor changes by $\mathbf{X}_t^M$ and take the function of portfolio losses subject to market risk of any particular bank in the system. This function can be written as

$$L_{t+1}^M = (f(t+1, \mathbf{Z}_t^M + \mathbf{X}_{t+1}^M) - f(t, \mathbf{Z}_t^M)) \tag{3.1}$$

If f is differentiable we can consider a first order approximation of the loss L_{t+1} of the form

$$L_{t+1}^{\Delta} = f_t(t, \mathbf{Z_t}) + \sum_{i=1}^{d} f_{z_i}(t, \mathbf{Z_t}) X_{t+1,i} \tag{3.2}$$

where the subscript to f denotes partial derivatives (see McNeil et al. [2005]).

This idea is applied to our problem as follows: Denote the price process of instrument i subject to market risk by $(S_{t,i})_{t\in\mathbb{N}}$. As risk factors we use logarithmic values, therefore $Z_{t,i} = \ln S_{t,i}, \quad i = 1,\ldots,d$. Risk factor changes $X_{t+1,i} = \ln S_{t+1,i} - \ln S_{t,i}$ then correspond to growth rates of the position values in the portfolio and hence

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{i=1}^{d} v_i S_{t,i}(\exp(X_{t+1,i} - 1). \tag{3.3}$$

where v_i denotes the value of position i. The linearized loss is then given by

$$L^{\Delta}_{t+1} = \sum_{i=1}^{d} v_i S_{t,i} X_{t+1,i} \tag{3.4}$$

The losses for all the banks due to market risk factor changes is then obtained by calculating the vector of linearized losses of all banks in the system in each scenario. Simulating many scenarios we therefore obtain the distribution of gains and losses of the sub portfolio of positions that are subject to market risk.

Chapter 4

The Credit Risk Model

For banks credit risk is relevant for all assets that have the nature of a debt contract. SRM distinguishes debt that is held between banks from debt held with parties outside of the banking system, such as loans to corporates or to households. While the value of interbank debt is determined in the network clearing model, the potential losses from non-interbank debt are captured by a credit risk model. The credit risk model used in SRM is a variant of the Credit Risk+ model (see CreditSuisse [1997]). While standard credit risk models are developed for the loan portfolio of a single institution, the credit risk model of SRM has to deal with a system of loan portfolios for different banks simultaneously.

From a risk analysis perspective the two major aspects of credit risk modeling is the distribution of the default frequency and the distribution of losses. While the distribution of default events contains information on whether the overall credit risk quality of the portfolios is improving or deteriorating, the loss distribution allows an assessment of the financial impact of the potential losses.

As in CreditRisk+, SRM makes no assumptions about causes of defaults. For each bank in the system there is exposure to default losses from a large number of obligors and the probability of default for each obligor is small. Such a situation is captured very well by the Poisson distribution. We consider the distribution of the number of default events over a quarter within portfolios of obligors having a range of different quarterly probabilities of default. The quarterly probability of default is determined by a mapping that combines individual rating information with industry sector default rates.

However default rates are not constant over time and have a high degree of variation. This fact has to be captured by the model. If default rate volatility is included the distribution of default events becomes skewed to the right with a significantly higher probability of extreme events while the expected number of defaults stays the same. In SRM the effects of default rate volatility are captured by combining the multivariate distribution of risk factor changes with an econometric model of default probabilities. This model "translates" macroeconomic risk factor changes to default probabilities for different industry sectors. Drawing macroeconomic risk factor changes many times from the multivariate distribution of risk factor changes allows us to incorporate the effects of default rate volatility.

Given the number of default events the distribution of default losses can be calculated of every bank in the system in every scenario. This distribution differs

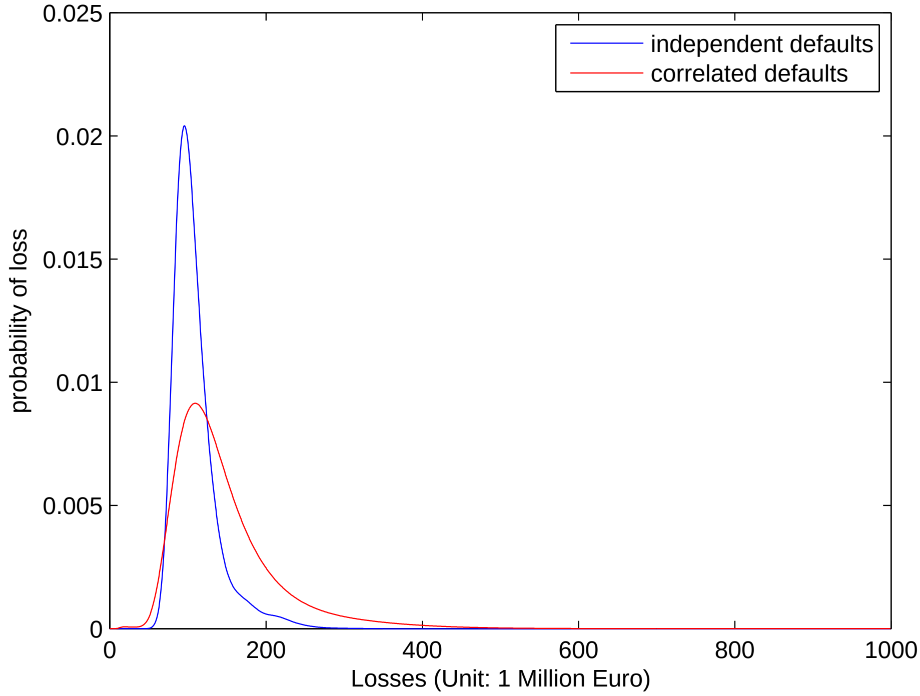

Figure 4.1. *The figure shows two loss distributions of a bank's loan portfolio under two different assumptions about default correlation. The blue curve is the loss distribution under the assumption that individual default events are Poisson distributed and independent. The red curve shows the loss distribution for the same portfolio assuming that individual default probabilities are themselves random variables and follow a probability distribution. Individual default events are assumed conditionally independent for each draw of individual default probabilities from this distribution. The ex ante loss distribution (red line) is skewed to the right and has fatter tails. This is the result of correlation of default events due to the volatility of individual default probabilities. The fact that individual default probabilities are high or low at the same time induces correlation in credit portfolio losses.*

from the distribution of default events because the amount of the loss depends on the exposure level of different obligors. The variation in exposure magnitudes results in a distribution that is not Poisson however the distribution is amenable to computation. The impact of considering default rate volatility is that the loss distribution has fatter tails than when default rate volatility is ignored. An example is shown in Figure 4.1. The blue density of losses is derived under the assumption that individual obligor defaults are independent. The red density function gives the density of losses under the same portfolio taking variation in obligor defaults rate into account.

4.1 From Macroeconomic Shocks to Industry Sector Probabilities of Default

Credit defaults occur due to risks that are idiosyncratic to the individual obligors. Often, however, there are background factors that cause default events to be correlated, even though there is no causal link between them. A unusually large number of defaults in a particular quarter might be the due to a beginning recession with defaults rates above their average long term level. Macroeconomic developments in general are the most influential factors behind default correlation and SRM therefore tries to capture the macroeconomic variables driving default rates.

To link the historical industry specific default rates to the macroeconomic variables we propose the model:

$$\mu_{i,t} = \frac{e^{X_t \beta_i}}{1 + e^{X_t \beta_i}} + \varepsilon_{i,t} \tag{4.1}$$

where we have $i = 1, \ldots, I$ industries, $\mu_{i,t}$ is the default rate of industry i at time t, X_t is a $1 \times k$ vector of macroeconomic risk factor changes, β_i is a $k \times 1$ vector of industry specific factor loadings, and $\varepsilon_{i,t}$ is a noise term. This functional specification is used in Boss [2002]. It takes into account that the default rate by definition can only take values between 0 and 1. For the calculation of consistent default rates we first calculate historic default rates $\mu_{i,t}$ for the industry sectors for the entire observation period. We define in a second step a set of admissible models and select the best one according to a specified criterion. We estimate the parameters β_i for the selected model ($\hat{\beta}_i$). Finally for the scenario generation we calculate the simulated default rates as

$$\hat{\mu}_{i,s} = \frac{e^{\hat{X}_t \hat{\beta}_i}}{1 + e^{\hat{X}_t \hat{\beta}_i}} \tag{4.2}$$

where $\hat{X}_s$ are the simulated macroeconomic risk factor changes in scenario s.[2]

These default rates can be interpreted as estimators of the probability of default of a loan in a particular industry sector in the given macroeconomic scenario $\hat{X}$. They are the conditional expectation of the default intensities in the different industry sectors. The conditioning variables are the macroeconomic risk factor changes that describe a particular macro scenario.

The data used for the estimations are the number of defaults and the number of firms in particular industry sectors as well as a set of macroeconomic variables. The first two variables are available at a more disaggregated level and are aggregated to a set of 11 larger sectors. The aggregated sectors are agriculture, forestry and mining (1), manufacturing (2), energy and water utilities (3), construction (4), whole sale and retail industry (5), hotel and restaurant industry (6), transport and telecommunication (7), banking and insurance industry (8), real estate industry (9), public services (10), and health industry (11). They are used to calculate the industry specific default probabilities. The reason why we aggregate is that the OeNACE classification system does not necessarily use categories that are the most useful from a credit risk viewpoint.

[2] Note that to arrive at the values $\hat{X}_t$ the draw of residuals from the multivariate risk factor distribution has to be combined with the macroeconomic VAR to calculate the appropriate log changes in macro variables for each scenario.

Variable	Abbreviation	Transformation
Dummy indicating the break in the year 1995	Y95	No transformation
Timetrend	Time	No transformation
Gross domestic product	GDP	Logs and first differences
Consumer Prices	CPI	Logs and first differences
Industrial production domestic	IPD	Logs and first differences
Industrial production international	IPI	Logs and first differences
Real effective exchange rate	XRATE	Logs and first differences
Oil prices	OIL	Logs and first differences
Domestic private debt	DEBT	Logs and first differences
Money	MONEY	Logs and first differences
Share prices	SHARES	Logs and first differences
Interest rate	IRATE	First differences
Inflation rate	InfRat	First differences
Disposable Income	IncVgr	Logs and first differences
IFO Index	IndExpIfo	Logs and first differences
Fixed investment	Inv	Logs and first differences
Investment in plant and equipment	InvEqu	Logs and first differences
Unemployment rate seasonally adjusted	UemRatSa	First differences
Real rate of return on 10 year bonds	Int10YRel	First difference
Dow Jones Industrials	Djx	Logs and first differences

Table 4.1. *Set of macroeconomic variables considered as candidates for the macro-econometric panel estimation of industry sector default intensities.*

Firm number data are collected from various sources. To get these data at a quarterly frequency we had to apply interpolation and extrapolation techniques, since number of firm data on the industry sector level are not usually available quarterly before 1997.

As a measure for historical default probabilities we use the actual default rate at an industry specific level. The default rate is the ratio between the number of defaults and the number of firms, such that

$$\mu_{i,t} = \frac{\text{Number of defaults}_{i,t}}{\text{Number of firms}_{i,t}}, \tag{4.3}$$

where i denotes the industry and t denotes the time period with data available at a quarterly frequency.

The set of macroeconomic variables that were considered as candidates is summarized in Table 4.1. These data are from the IMF Financial Sector Statistics, from the OECD from the Austrian Central Bank (OeNB) and from the Austrian Institute of Economic Research (WIFO).

The estimation technique is based on a paper by Papke and Wooldrige [1996] and is described in the technical appendix.

We estimate model (4.1) for the 11 industry sectors that we have aggregated from the OeNACE sectors. The dependent variable is the default rate and the exogenous variables are drawn from a pool of macroeconomic variables. This pool includes 19 variables.

To find the most parsimonious model we pursue the procedure from "simple to general". We start with models that contain only one right hand side variable and go on to models with more variables. All combinations of macroeconomic variables are estimated and the best model is taken. However, to save computer time the number of possible combinations of variables on the right hand side is restricted to four. Thus, we select at most four variables out of 19.

To select to best model for each industry sector, the likelihood and the Bayesian information criterion (BIC) are used. The first model selection is done by finding the model with the largest likelihood; the second by finding the model with the smallest BIC. In the implementation of SRM we use the model selected based on the data of September 2005.

Add some examples, pictures and typical graphics

4.2 From Industry Default Probabilities to Loss Distributions

Once we have determined obligor default probabilities, we follow closely the Credit Risk+ framework. So far we have only developed a model to determine default intensities at the level of 11 aggregate industry sectors. Applying these default intensities as estimators of the probability of default of a loan in a particular industry would imply that we treat every loan in – say construction – as equally risky irrespective of the borrower to which this loan has been extended. This is a broad brush of corporate obligor's risk assessment. Since we have also individual rating information in the central loan register we can adjust the estimate of the industry level default intensity by the rating information. Denote by μ_i^R the probability of default of an obligor based on his rating and let $\hat{\mu}_i$ the default intensity for a particular macroeconomic scenario. If μ_{i0} is the empirical default rate in industry i in the observation period, we write the estimator of the default probability of an individual obligor as

$$\mu_i = \mu_i^R \frac{\hat{\mu}_i}{\mu_{i0}} \tag{4.4}$$

Based on these values μ_i we proceed as in CreditSuisse [1997]. The technical derivation is described in the appendix. The general idea of the approach is to develop a model for the distribution of default events and for credit losses that is amendable to efficient computation, while taking into account correlating factors of default events. The idea can perhaps be best described graphically and is shown in 4.2.

In the simulation this recursion is used to build a loan loss distribution for each bank. From this distribution a loss is drawn independently for all banks. To save computing time we draw for a given macro scenario and thus for a given system of loan loss distributions more than one credit loss profile. The corresponding market risk losses are combined with these credit risk losses by taking market return draws from the conditional risk factor change distribution. For example we could generate 1000 macro-scenarios and take 1000 draws from the credit loss distributions for each macro-scenarios and combine these draws with 1000 draws from the conditional distribution of market risk factor changes we generate 1.000.000 different loss scenarios.

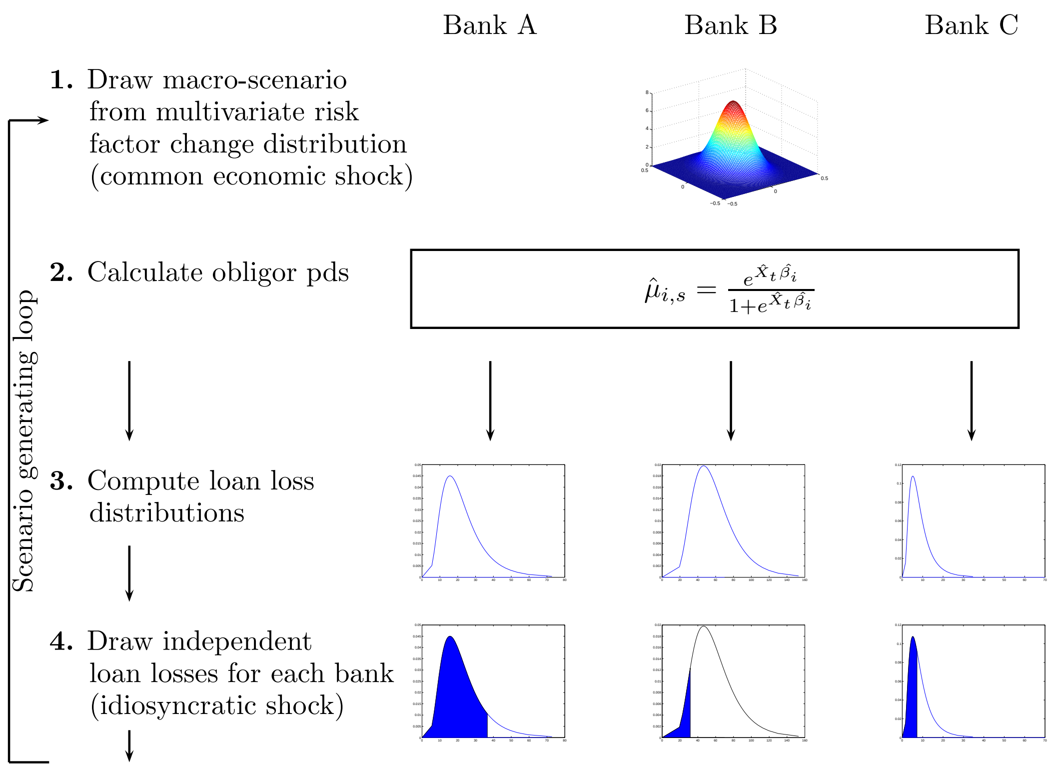

Figure 4.2. *Macroeconomic scenarios are part of the simulation draws from the multivariate distribution of risk factor changes. Given this draw obligor default probabilities can be determined and loss distributions are calculated. Losses are drawn for this scenario independently from these loss distributions. Then a new scenario is drawn.*

Chapter 5

The Network Model

The simultaneous analysis of market and credit risk in SRM is combined with a detailed model of interbank relations. This network model both adds a detailed picture of bilateral financial relations among domestic banks, both in terms of debt and equity, and allows to close the model by a system wide clearing procedure. The clearing procedure gives a precise answer to the question under which conditions the financial claims in the banking system are consistent with the promises implicit in the complex network of financial contracts ex ante and after uncertainty has been resolved.

What does consistency mean in this context? Ex ante banks are related by financial promises that require them to pay or entitle them to receive certain payments after uncertainty has unfolded and risk factor changes have been realized. In an adverse scenario this may imply that given these (past) promises and this (current) realization of risk factor changes the total net value of one or more institutions is negative. In this case banks with a negative value are insolvent. The value of shares others hold in these bank is zero in such a case and the value of debt is whatever creditors can realize from the remaining asset value under a certain sharing rule. This potentially reduced value of claims may induce further insolvencies of banks that would not have been insolvent under more favorable risk factor changes in a second round effect. The clearing procedure works out this value adjustment process for each realization of risk factor changes. For a complex network of interbank debt and equity claims working out this clearing procedure is a nontrivial task.

In this chapter we describe the network model in an informal way, we introduce some notation and concepts and illustrate them by way of toy examples. A technical description containing concepts and proofs is in the appendix.

5.1 Describing the banking system

Our network model of interbank credits and share holdings is an extended version of the model of Eisenberg and Noe [2001]. The inclusion of interbank share holdings is an innovation of SRM and allows a fully fledged network analysis of financial interbank relations.

We denote the set of banks by $\mathcal{N} = \{1, ..., n\}$. Between the banks we have debt and equity claims. Interbank debt is described by a matrix $L \in \mathbb{R}_+^{n \times n}$ of nominal interbank liabilities. We use the convention that L_{ij} denotes the nominal liabilities of bank i toward bank j. Since the interbank liabilities of bank i are at the same time the interbank assets of bank j we see the values of interbank liabilities in the rows and the values of interbank assets in the columns of matrix L. We have also a matrix $\Theta \in [0,1]^{n\times n}$ of (outside) shareholdings between banks. Θ_{ij} denotes the share that bank i holds in bank j, so that

$$\sum_{i=1}^{n} \Theta_{ij} \leq 1 \tag{5.1}$$

The specification does allow for the case that a bank is among the outside shareholders of its own shares ($\Theta_{ii} < 1$). The only restriction we impose on the interbank shareholdings is that there is no group of banks in which *each* bank is completely owned by other banks in that group, in particular $\Theta_{ii} < 1$. This is summarized in the following

Assumption There exists no subset $\mathcal{I} \subset \{1,\ldots,n\}$ such that

$$\sum_{i\in\mathcal{I}} \Theta_{ij} = 1 \quad \text{for all} \quad j \in \mathcal{I}.$$

Θ is called a *holding matrix* if it fulfills this assumption.

Banks hold not only financial positions among each other but also with other parties, outside of the banking system. These financial positions and the risk of their value changes are described by the market and credit risk model of SRM. In the network model these positions are considered as the endowment of a bank with an exogenous income position $e_i \in \mathbb{R}$. Note that this position also includes shareholdings in other companies than banks.

In each state the banking system is thus completely described by four objects: The interbank liability matrix, the interbank share holding matrix, and the income position e, (L,Θ,e). These are the parameters we try to identify form the data in the initial period. Risk factor changes will initially change the value of e and at the same time through the clearing procedure also the value of interbank and non interbank debts and shares as well as other interbank assets.

Let us illustrate the concepts introduced by an example: Consider a system with three banks. The inter-bank liability structure is described by the matrix

$$L = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Bank 3 has liabilities of 3 with bank 1 and liabilities of 1 with bank 2. It has of course no liabilities with itself. Reading this matrix row wise gives us the value of interbank assets. Assume bank 1 holds 30% in banks 2 and 3, bank 2 holds 10% in bank 3 and bank 3 holds 20% in bank 1 and 10% in bank 2. This would give a matrix Θ as

$$\Theta = \begin{pmatrix} 0 & 0 & 0.2 \\ 0.3 & 0 & 0.1 \\ 0.3 & 0.1 & 0 \end{pmatrix}$$

Lets assume that the net income position of other activities outside of interbank relations is summarized by the vector $e = (1,2,6)$. To decide whether this system of financial promises is consistent, we need a clearing procedure.

5.2 Clearing

A bank is insolvent whenever its net income position form non interbank business plus the amounts received from other banks are insufficient to cover its liabilities. In case of default the clearing procedure has to respect three criteria:

1. limited liability, which requires that the total payments made by a bank must never exceed the cash flow available to the bank,
2. priority of debt claims, which requires that stockholders in the bank receive no value unless the node is able to pay off all of its outstanding debt completely, and
3. proportionality, which requires that in case of default all creditors are paid off in proportion to the size of their claim on firm assets.

To operationalize proportionality let $\bar{p}_i$ be the total nominal obligations of bank i,

$$\bar{p}_i = \sum_{j=1}^{n} L_{ij} + D_i \tag{5.2}$$

and define the proportionality matrix Π by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

Evidently, it has to hold that $\Pi \cdot \vec{1} \leq \vec{1}$ where $\vec{1}$ is an $n \times 1$ vector of ones. Let $p = (p_1, \ldots, p_n)' \in \mathbb{R}^n_+$ be the *actual* payments made by banks to its interbank and non interbank creditors under the clearing mechanism. Let $V \geq \vec{0}$ be a vector of equity values and define the map

$$\Upsilon(V, p, e, \Pi, \Theta) = [e + \Pi' p - p + \Theta V] \vee \vec{0} \tag{5.4}$$

For a given vector p a consistent vector of equity values $V^*(p)$ is a fixed point of $\Upsilon(\cdot; p, e, \Pi, \Theta) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$V^*(p) = [e + \Pi' p - p + \Theta V^*(p)] \vee \vec{0} \tag{5.5}$$

where $\vec{0}$ denotes the $n \times 1$ dimensional zero vector. In the Appendix we give a proof which establishes that there exists a unique fixed point, $V^*(p)$. A vector of actual payments p^* that respects the clearing criteria,

$$p_i^* = \begin{cases} 0 & \text{if} & e_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \leq 0 \\ e_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) & \text{if} & 0 \leq e_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \leq \bar{p}_i \\ \bar{p}_i & \text{if} & \bar{p}_i \leq e_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \end{cases}$$

is called a clearing payment vector. This can be summarized in a more compact notation in the following

Definition 5.1. *A vector $p^* \in [\vec{0}, \bar{p}]$ is a* **clearing payment vector** *if and only if*

$$p^* = \left\{ [e + \Pi' p^* + \Theta V^*(p^*)] \vee \vec{0} \right\} \wedge \bar{p} \tag{5.6}$$

where $V^*(p^*)$ is the unique solution of Equation 5.5.

Technically a clearing vector p^* is a fixed point of the map $\Phi(\cdot;\Pi,\bar{p},e,\Theta)$: $[\vec{0},\bar{p}] \rightarrow [\vec{0},\bar{p}]$ defined by

$$\Phi(p;\Pi,\bar{p},e,\Theta) = \left\{[e + \Pi' p + \Theta V^*(p)] \vee \vec{0}\right\} \wedge \bar{p} \tag{5.7}$$

In the appendix we show that under our assumptions on (L, Θ, e) and the clearing procedure clearing payment vectors always exist and are unique. Furthermore we present an algorithm we can use to calculate clearing payment vectors.

5.3 Analyzing clearing payment vectors

From the solution of the clearing problem, we can gain not only information about insolvencies but also with respect to systemic stability. Default of bank i is called fundamental if bank i is not able to honor its promises under the assumption that all other banks honor their promises A contagious default occurs, when bank i defaults only because other banks are not able to keep their promises. This distinction allow us to analyze defaults that result directly from risk factor movements form defaults that indirectly result from second round effects of insolvency contagion through interbank relations.

Insert toy example about here

5.4 Simulation

As there is no closed form solution for the distribution of p^*, given the distribution of e, we have to resort to a simulation approach where each draw is called a scenario. We know that there exists a (unique) clearing payment vector p^* for each scenario. Thus from an ex-ante perspective we can assess expected default frequencies from inter-bank credits across scenarios as well as the expected severity of losses from these defaults given that we have an idea about the distribution of bank values V.

5.5 Estimation of bilateral interbank loans

Insert updated description of L estimation here

5.6 Extensions

In the appendix we add two additional features to the network model. The first feature is that we allow for a richer structure of seniority between different forms of debt. In the discussion of the network model presented here we have treated all debts as if they belonged to the same seniority class. Second we add bankruptcy costs to to picture. In the version of the model we have presented here we have always assumed that in the case of default the full value of the insolvent institution is proportionally transferred to the creditors. In reality part of the value will be lost in insolvency and sometimes all interbank payments will be stopped immediately. With the inclusion of bankruptcy costs we can analyze models with these more realistic assumptions.

Chapter 6

Data

6.1 Data Input

SRM uses two different categories of data as input:

- Panel-data reported by Austrian banks to the OeNB reflecting on- and off-balance sheet positions of banks at the end of a pre-specified quarter and other relevant for supervisory information purposes like regulatory capital.
- Time-series data from internal external sources reflecting historical movements of market-, credit and macroeconomic riskfactors.

6.1.1 Data reported by Austrian banks to OeNB

Regarding this data category the following reports are used in SRM on a bank by bank basis

- MAUS-A (Monatsausweis, Teil A, Geschäftsstrukturdaten): Part A of the monthly report of Austrian banks covering the asset and the liability structure of the balance sheet as well as information on off balance sheet items.
- MAUS-B2 (Monatsausweis, Teil B2, Zinsrisikostatistik): The quarterly reported statistic on interest rate risk, which provides a breakdown of all interest rate sensitive on- and off-site balance sheet assets and liabilities with respect to their repricing maturity, i.e. each instrument is reported with respect to the maturity bucket within which the next fixing of the respective interest rate takes place.
- MAUS-C (Monatsausweis, Teil C, Aufsichtsrelevante Zusatzdaten): Part C of the monthly report, which covers additional supervisory information, in particular on capital requirements and net open positions in foreign currencies.
- Data on the structure of participations among Austrian banks, which is calculated by the OeNB on the basis of banks reports on their direct and indirect participations in other financial and non-financial institutions
- GKE (Grosskreditevidenz): The central Austrian register on large credit exposures, which covers credit risk sensitive instruments with a volume of more

than EUR 350.000 on a customer by customer basis. It includes the outstanding volume of securitized and non-securitized loans, guarantees and commitments as well as respective collaterals, specific provisions and the internal rating of the customers credit quality.

- Basic data on banks like the identification code, the name, the zip code, the sector according to the official breakdown of the Austrian banking sector, etc
- Basic data on the debtors included in the GKE, like an (anonymous) identification code, the industry sector according the NACE-classification, etc.
- Basic data on the positions reported by banks, like their identification code, the respective risk category, etc.

Given B banks and D debtors[3] and P^M positions reported by banks to MAUS and P^G positions reported to GKE we have the following input data:

- A matrix of panel data of dimension B x P^M
- An array of panel data of dimension B x D x P^G
- A vector of information on banks of length B
- A vector of information on banks of debtors of length D

6.1.2 Time Series from External Data Sources

This category includes the following

- Macro-economic time series on a quarterly basis from the OeNBs macroeconomic database starting with the first quarter of 1969.
- Time series for market risk factors from Bloombergs financial data services with the first quarter of 1980.
- Quarterly Time series on the number of insolvencies and the total number of Austrian firms according to the NACE-classification (OeNace Level2) starting with the first quarter of 1969.
- Basic data on the time series, like an identification code, the respective risk category, etc

Given S time-series in sum and Q quarters of observations in time we have the following input data:

- A matrix of time series data of dimension Q x S.
- A vector of information on time series of length S.

6.2 Building Risk Positions from Banks Report Data

In this section it will be described how the data provided by Austrian banks through various reports is used within SRM to build up risk and capital positions

[3]With B=845 and D=70.000 as of end December 2005, where B does not include branches of foreign banks in Austria (internal type equals ZW), which can be included in SRM optionally, but are excluded by default.

6.2.1 Positions exposed to market risk

Positions exposed to foreign exchange rate risk. For the net open positions in foreign currencies SRM uses the peaks of net open positions within the last month referred to the reporting date. This is an exception to the rule of SRM, which in general requires the analysis to be based on positions at the end of each quarter. However, as no such data is available and because the actual risk at the end of each quarter will at most be overestimated, the peaks of net open positions are used.

The respective positions are reported by banks directly with respect to 12 major currencies and include on-balance as well as off-balance sheet items. For the other currencies, reporting is on a voluntary basis. As no bank in our sample has open positions of more than 1% of total assets in any other reported currency, currently in SRM exchange rate risk is considered with respect to USD, CHF, JPY, and GBP.

It should be noted, that reporting with respect to currencies of Central and Eastern European countries is not yet mandatory. As some Austrian banks are highly active in these countries respective open currency positions could be relevant in terms of risk. However, reporting of open positions in these currencies will be mandatory in the future and the respective data will be included in SRM as soon as it is available.

Positions exposed to interest rate risk Data on interest rate sensitive positions are reported by banks within the statistics on interest rate risk. It covers all interest rate sensitive on-balance and off-balance sheet items. In general, the statistics includes banking as well as trading book positions. However, banks running a "large" trading book (according to the Banking Act) do not report their trading book positions in this statistics (as these positions are subject to own funds requirements for interest rate risk according to the Capital Adequacy Directive). The data represent book values of the interest rate sensitive positions. The positions are slotted into 13 time buckets according to the time to next interest rate adjustment of a position (time to re-pricing). Assets and liabilities are reported with respect to these maturity buckets separately for EUR, USD, JPY, GBP, CHF, and a residual representing all other currencies. On the basis of this information we calculate net positions in the available currencies - neglecting the residual - with respect to four different maturity buckets: up to 6 months, 6 months to 3 years, 3 to 7 years, more than 7 years. For the valuation of net positions in these maturity buckets we use the 3 month, 1 year, 5 years and 10 years interest rates in the respective currencies. The number of maturity buckets was reduced from 13 to four in order to reduce the number of risk-factors necessary to value the respective positions.

Positions exposed to equity price risk. With respect to equity price risk the available data are with respect to on-balance sheet positions only. Equity data are structured by the legal definition of the equity share (participation, share in affiliated company, other), by the fact whether the equity is listed on a stock exchange or not and by origin into domestic and foreign. On the basis of this data domestic and foreign positions exposed to equity price risk are calculated where only equity listed on a stock exchange is included.

Code	Net open position in	Riskfactor
Fx.Chf	CHF	Exchange Rate EUR/CHF
Fx.Gbp	GBP	Exchange Rate EUR/GBP
Fx.Jpy	JPY	Exchange Rate EUR/JPY
Fx.Usd	USD	Exchange Rate EUR/USD
Equity.NonDom	Non-domestic equity	Domestic equity index ATX
Equity.Dom	Domestic equity	MSCI-World-Index
IntRate.Eur.00Y03M00D	EUR up to 6 month	3-month interest rate in EUR
IntRate.Eur.01Y00M00D	EUR between 6 month and 3 years	1-year interest rate in EUR
IntRate.Eur.05Y00M00D	EUR between 3 and 7 years	5-years interest rate in EUR
IntRate.Eur.10Y00M00D	EUR of more than 7 years	10-years interest rate in EUR
IntRate.Usd.00Y03M00D	USD up to 6 month	3-month interest rate in USD
IntRate.Usd.01Y00M00D	USD between 6 month and 3 years	1-year interest rate in USD
IntRate.Usd.05Y00M00D	USD between 3 and 7 years	5-years interest rate in USD
IntRate.Usd.10Y00M00D	USD of more than 7 years	10-years interest rate in USD
IntRate.Chf.00Y03M00D	CHF up to 6 month	3-month interest rate in CHF
IntRate.Chf.01Y00M00D	CHF between 6 month and 3 years	1-year interest rate in CHF
IntRate.Chf.05Y00M00D	CHF between 3 and 7 years	5-years interest rate in CHF
IntRate.Chf.10Y00M00D	CHF of more than 7 years	10-years interest rate in CHF
IntRate.Jpy.00Y03M00D	JPY up to 6 month	3-month interest rate in JPY
IntRate.Jpy.01Y00M00D	JPY between 6 month and 3 years	1-year interest rate in JPY
IntRate.Jpy.05Y00M00D	JPY between 3 and 7 years	5-years interest rate in JPY
IntRate.Jpy.10Y00M00D	JPY of more than 7 years	10-years interest rate in JPY
IntRate.Gbp.00Y03M00D	GBP up to 6 month	3-month interest rate in GBP
IntRate.Gbp.01Y00M00D	GBP between 6 month and 3 years	1-year interest rate in GBP
IntRate.Gbp.05Y00M00D	GBP between 3 and 7 years	5-years interest rate in GBP
IntRate.Gbp.10Y00M00D	GBP of more than 7 years	10-years interest rate in GBP

Table 6.1. *Positions. Times are times to maturity*

The procedure described above gives us a vector of 26 market risk exposures, 4 foreign currency, 2 equity, and 20 interest rate (4 maturities for 5 currencies), risk exposures for each bank. Thus we get a $B \times 26$ matrix of market risk exposures with respect to the following risk factors summarized in Table 6.1.

6.2.2 Positions Exposed to Credit Risk

To analyze credit risk we use in addition to the data provided by MAUS the major loans register of OeNB (GKE) which provides detailed information on the banks' loan portfolios to non-banks. This database contains securitized and non-securitized loans as well guarantees and other instruments affected by credit risk to domestic and foreign customers and banks exceeding a volume of 350.000 Euro on a borrower by borrower basis. Outstanding volumes are reported as well as credit lines. However, there are two additional exceptions regarding the reporting to the GKE. First, loans to the central and regional governments are exempted from reporting in general and second, interbank loans referring to short term interbank transactions are also not required to be reported [4].

In addition to instruments affected by credit risk itself, for each borrower the sum of collaterals and eventual specific provisions regarding the outstanding volume

[4]The GKE database covers about two third of all loans of Austrian banks in terms of nominal values.

are reported. On the basis of the different instruments and collaterals reported to the GKE the net credit risk exposure of each bank with respect to each borrower is calculated as follows.

Net credit risk exposure equals

- outstanding volume of non-securitized loans (nonrevolving, revolving, and trustee loans, exchange bills and claims related to leasing activities) plus
- outstanding volume of securitized loans plus
- non-exercised volume of credit lines regarding non-securitized loans plus
- other commitments affected with credit risk plus
- outstanding volume of granted guarantees affected with credit risk plus
- non-exercised volume of credit lines regarding granted guarantees

less the maximum of

- outstanding volume guarantees provided by the government plus non-exercised volume of credit lines regarding government guarantees and
- collaterals

According to the reporting rules government guarantees should be included in collaterals. However, in some cases this is obviously not the case, as reported government guarantees exceed reported collaterals. Hence, it was decided to use the maximum of both positions in order to determine total collaterals. Depending on whether specific provisions are considered in the capital positions or not (see section 6.5), specific provisions reported for each borrower are also subtracted in cases where provisions are not included in capital.

In order to cover loans below the reporting threshold of GKE, we use a statistic included in MAUS-A, which provides the number of loans to domestic customers according to the following buckets: up to 10.000 EUR, 10.001 to 50.000 EUR, 50.001 to 100.000 EUR, 100.001 to 500.001 EUR, 500.001 to 1.000.000 EUR, 1.000.001 to 3.000.000 EUR and above 3.000.000 EUR. These buckets are used to build up the loan portfolio to domestic customers below the reporting threshold of GKE for each bank, where the following buckets are considered: up to 10.000 EUR, 10.001 to 50.000 EUR, 50.001 to 100.000 EUR, 100.001 to 349.999 EUR. The number of loans in the last of these buckets is calculated as the number of loans to domestic customers reported according to MAUS-A in the bucket 100.001 to 500.001 EUR minus the number of loans to domestic customers with a volume up to 500.001 EUR reported to GKE.

Finally, each bank reports its internal rating for each borrower to the GKE. Within the OeNB the individual ratings provided by banks are mapped on a general master scale. There are two versions of the OeNB master scale. The rough version this master scale provides 7 non-default rating classes and 6 default rating classes, while the precise version provides 21 non-default rating classes and six default rating classes. However, in many cases, in particular with small banks, only the rough mapping is available. The master-scales allow the assignment of a certain probability of default for each rating class according to Table 6.2.

However, in cases were a certain borrower is rated by more than one bank, because he has loans from more than one bank, the borrowers rating could be

Code	Type	Probability of Default (Pd)		
		Average	Lower Limit	Upper Limit
1	rough	0.0020%	0.0010%	0.0040%
2	rough	0.0100%	0.0040%	0.0220%
3	rough	0.0490%	0.0220%	0.1090%
4	rough	0.2440%	0.1090%	0.5460%
5	rough	1.2210%	0.5460%	2.7310%
6	rough	6.1060%	2.7310%	13.6530%
7	rough	30.5280%	13.6530%	68.2630%
8.1	rough	100.0000%	100.0000%	100.0000%
8.2	rough	100.0000%	100.0000%	100.0000%
8.3	rough	100.0000%	100.0000%	100.0000%
8.4	rough	100.0000%	100.0000%	100.0000%
8.5	rough	100.0000%	100.0000%	100.0000%
8.6	rough	100.0000%	100.0000%	100.0000%
10	precise	0.0011%	0.0009 %	0.0015%
20	precise	0.0020%	0.0015%	0.0026%
30	precise	0.0033%	0.0026%	0.0044%
40	precise	0.0057%	0.0044%	0.0075%
50	precise	0.0098%	0.0075%	0.0128%
60	precise	0.0167%	0.0128%	0.0218%
70	precise	0.0286%	0.0218%	0.0374%
80	precise	0.0488%	0.0374%	0.0639%
90	precise	0.0835%	0.0639%	0.1092%
100	precise	0.1428%	0.1092%	0.1868%
110	precise	0.2442%	0.1868%	0.3194%
120	precise	0.4176%	0.3194%	0.5461%
130	precise	0.7141%	0.5461%	0.9338%
140	precise	1.2211%	0.9338%	1.5968%
150	precise	2.0881%	1.5968%	2.7305%
160	precise	3.5706%	2.7305%	4.6692%
170	precise	6.1057%	4.6692%	7.9841%
180	precise	10.4406%	7.9841%	13.6527%
190	precise	17.8531%	13.6527%	23.3458%
200	precise	30.5284%	23.3458%	39.9207%
210	precise	52.2028%	39.9207%	68.2635%
220.1	precise	100.0000%	100.0000%	100.0000%
220.2	precise	100.0000%	100.0000%	100.0000%
220.3	precise	100.0000%	100.0000%	100.0000%
220.4	precise	100.0000%	100.0000%	100.0000%
220.5	precise	100.0000%	100.0000%	100.0000%
220.6	precise	100.0000%	100.0000%	100.0000%

Table 6.2. *The OeNB Master Scale*

different for different banks and it is not obvious, which rating should be assigned to this specific borrower. Hence SRM provides various options in order to deal with these cases.

- Reported: assigns to each loan the rating assigned by the bank. Hence a borrower specific rating can be different depending on the bank granting the loan.
- Maximum: assigns always the worst rating reported by one bank for the

specific borrower.

- Mean: assigns always the mean rating (correct: the mean respective Pd), reported by one bank for the specific borrower.

The riskiness of an individual loan to domestic customers is assumed to be characterized by two components: the rating which is assigned by the bank to the respective customer and allows us to assign a probability of default to each borrower according to the table given above, and the default frequency of the industry sector the customer belongs to.

The default frequency data are from the Austrian rating agency Kreditschutzverband (KSV). The KSV database provides us with time series of insolvencies[5] and the total number of firms in most NACE branches at a quarterly frequency starting in 1969. This allows us to calculate a time series of historically observed default frequencies for the 13 industry sectors described by dividing the number of insolvencies by the number of total firms for each industry sector and quarter.

Starting with the first quarter of 1997 the number of insolvencies per quarter and industry sector is provided directly from the KSV database. However, before this date the respective information was only available on paper. Hence, for the present project this information had to be preprocessed in order to get a time series of the number of insolvencies per quarter and industry sector starting from the first quarter of 1969 to the last quarter of 1996. This was done on the basis of weekly insolvency reports, which were available on paper only and provided to the SRM project team. The weekly lists were then scanned into the computer and preprocessed by an OCR-software in order to produce text files of the weekly insolvency list. Finally these lists were processed semi-automatically in order to assign a NACE code to each case of insolvency. Finally this procedure resulted in a quarterly time-series of the number of insolvencies in each industry sector for the period of the first quarter of 1969 to the last quarter of 1996, which could the be connected with the data provided by the KSBV data base.

The time series of default frequencies is explained by macroeconomic risk factor changes using an econometric model described in chapter 4. By this estimated equation we can translate macroeconomic risk factor changes in probabilities of default for each industry branch. These default probabilities serve beside the default probabilities provided through GKE - as a second input to the credit risk model. To construct insolvency statistics for the private and the residual sector, where no reliable information on number of insolvencies and sample size is available, we take averages from the data that are available.

We assign the domestic loans to non-banks to 13 industry sectors (basic industries, production, energy, construction, trading, tourism, transport, financial services, public services, other services, health, private households, and a residual sector) based on the NACE-classification of the debtors. Furthermore we add regional sectors (Western Europe, Central and Eastern Europe, North America, Latin America and Carribean, Mid East, Asia and Far East, Pacific, Africa, and a residual sector) for foreign banks and non-banks individually, resulting in a total of 18 non-domestic sectors. Since only loans above a threshold volume are reported to the GKE we assign domestic loans below this threshold to the domestic residual

[5]Insolvencies refer to the opening of bankruptcy proceedings as well as to dismissals of bankruptcy filings for lack of assets. The number of enterprises per quarter and industry sector was estimated on the basis of data provided by the Association of Austrian Social Security Institutions, Statistics Austria and the KSV.

Code	Credit Exposure to	Source of Sector
Dom.NonBanks.Bas	Domestic non-banks , basic industries	Macro-model
Dom.NonBanks.Prod	Domestic non-banks , production	Macro-model
Dom.NonBanks.Engy	Domestic non-banks , energy	Macro-model
Dom.NonBanks.Cstr	Domestic non-banks , construction	Macro-model
Dom.NonBanks.Trad	Domestic non-banks , trading	Macro-model
Dom.NonBanks.Tour	Domestic non-banks , tourism	Macro-model
Dom.NonBanks.Trsp	Domestic non-banks , transport	Macro-model
Dom.NonBanks.Fin	Domestic non-banks , financial services	Macro-model
Dom.NonBanks.SrvC	Domestic non-banks , public services	Macro-model
Dom.NonBanks.SrvP	Domestic non-banks , other services	Macro-model
Dom.NonBanks.Hlth	Domestic non-banks , health	Macro-model
Dom.NonBanks.Priv	Domestic non-banks , private households	Macro-model
Dom.NonBanks.Oth	Domestic non-banks , residual sector	Macro-model
Dom.Banks.	Domestic banks (not included in the network model)	Average rating
NonDom.NonBanks.Europe.West	Foreign non-banks, Western Europe	Average rating
NonDom.NonBanks.Pacific	Foreign non-banks, Pacific	Average rating
NonDom.NonBanks.America.North	Foreign non-banks, North America	Average rating
NonDom.NonBanks.Asia.East	Foreign non-banks, Asia and Far East	Average rating
NonDom.NonBanks.America.South	Foreign non-banks, Latin America and Carribean	Average rating
NonDom.NonBanks.Asia.West	Foreign non-banks, Mid East	Average rating
NonDom.NonBanks.Africa	Foreign non-banks, Africa	Average rating
NonDom.NonBanks.Europe.East	Foreign non-banks, Central and Eastern Europe	Average rating
NonDom.NonBanks.Oth	Foreign non-banks, Residual, Int. Org. and Unallocated	Average rating
NonDom.Banks.Europe.West	Foreign banks, Western Europe	Average rating
NonDom.Banks.Pacific	Foreign banks, Pacific	Average rating
NonDom.Banks.America.North	Foreign banks, North America	Average rating
NonDom.Banks.Asia.East	Foreign banks, Asia and Far East	Average rating
NonDom.Banks.America.South	Foreign banks, Latin America and Carribean	Average rating
NonDom.Banks.Asia.West	Foreign banks, Mid East	Average rating
NonDom.Banks.Africa	Foreign banks, Africa	Average rating
NonDom.Banks.Europe.East	Foreign banks, Central and Eastern Europe	Average rating
NonDom.Banks.Oth	Foreign banks, Residual, Int. Org. and Unallocated	Average rating

Table 6.3. *The Sector Classification*

sector. This is done on the basis of a report that is part of MAUS and provides the number of loans to domestic non-banks with respect to different volume buckets. For non-domestic loans no comparable statistic is available. However, one can assume that most of cross-border lending exceeds the threshold of 350.000 Euro and hence the associated risk can be neglected.

Default probabilities for the non-domestic sectors are calculated as averages of the default probabilities according to the ratings that are assigned by all banks to all customers within a given foreign sector. Finally, we add one sector for domestic banks, which are for some reason are not included in the network (SRM optionally allows to restrict the set of banks included into the network according to some criteria, like total assets). Default Probabilities for this sector are calculated in the same way as for non-domestic sectors. Thus we get a $B \times 32$ matrix of credit risk exposures with the following sources of respective default probabilities:

6.3 Estimation of the matrix of interbank liabilities

As interbank loans and liabilities are partially reported on a bank by bank basis the matrix describing the liability structure for all Austrian banks has to be estimated on the basis of the available data. This is done using entropy maximization, which is presented in detail in chapter 5 and in the appendix. In the following this will be described in some detail from the perspective of the data defining the constraints used by entropy maximization. In addition the available data can be used to set some of the entries to zero

Constraints on single entries of the interbank matrix. There are two sources, which provide us with information on single entries of the interbank matrix. First, securitized and non-securitized loans exceeding a volume of EUR 350.000 are reported in the central credit register on a bank by bank basis. Second, banks belonging to one of the hierarchically structured Austrian banking sectors (Raiffeisen, Savings Banks and Volksbanken) report non-traded and Euro-denominated loans and liabilities to their respective head institute within their monthly reports. It should be noted that neither of the two data sources covers all potential loans and liabilities between two specific banks. Regarding the central credit register, the volume not captured due to the reporting limit could be negligible, but as loans due to short term interbank transactions are not reported to the central credit register, a potentially large part of loans will be missed. The data from the monthly report however only refers to loans and liabilities, which are not traded and denominated in Euro. Hence as a constraint it will be required the respective entry in the matrix hast to be equal or greater than the maximum reported by the the two data sources

Constraints on the sum of columns of the interbank matrix. The sum over all columns of the interbank matrix is a row vector which elements can be interpreted as the total volume of loans granted by a bank specified by the rows of to all other banks. This information is reported by Austrian banks through the positions loans to domestic banks and bonds issued by domestic bank. Hence the sum of the columns can be constrained to be equal the sum of these two positions.

Constraints on the sum of rows of the interbank matrix. The sum over all rows of the interbank matrix is a column vector which elements can be interpreted as the total volume of liabilities of a bank specified by the rows with respect to all other banks. This information is not fully provided by Austrian banks, as the reported position called liabilities to domestic banks refers only liabilities, mostly deposits, which are not traded on a stock exchange. There is no equivalent regarding bonds for the sum of rows, as the issuing bank of a bond cannot know which part of the issued volume is hold by domestic banks. However, a bank reports the sum of all bonds it has issued. Hence the sum of the rows is constrained to be equal or greater than the liabilities to all domestic banks and equal or less than the liabilities to domestic banks plus the issued bonds.

Restrictions on Sub-Matrices. Finally we have information on sub matrices through data provided by MAUS. Each bank reports the loans and liabilities, which are not traded and denominated in Euro, with respect to each of the seven sectors of the Austrian banking system. These sectors are: joint stock banks, savings banks, state mortgage banks, Raiffeisen banks, Volksbanken, construction savings and loans, special purpose banks. As this information refers only to some part of total loans and liabilities to each of the sector, the sum of the columns representing the banks of a certain sector can be constrained to be equal or larger than the respective reported loans to this sector. Analogously, the sum of the rows representing the banks of a certain sector can be constrained to be equal or larger than the respective reported liabilities to this sector.

In practice, the algorithm described in the appendix is used to estimate the interbank matrix as follows. Because the restrictions described above, are inconsistent to some extent, in a first step all constraints on the matrix as described above are implied to the algorithm. However, due to the inconsistencies in the data, the algorithm does not converge for this set of constraints. Hence in a second step the algorithm is restarted neglecting the constraints of category four using the result from the first step as a prior. Now convergence can easily be achieved. However, in cases were only few information regarding single entries of the matrix is available, the algorithm tends to assign a quite low amount of loans and liabilities to a relatively large number of counterparty banks, which in reality may have no interbank relation to that specific bank at all. In order to avoid this, all connections below a certain level (i.e. 10,000 EUR) are cut down to zero after convergence was achieved. Using the resulting matrix as a prior, the algorithm is then restarted until convergence is achieved again. This procedure is repeated until no further connections are cut down after convergence was achieved.

Figure 6.1 shows the regional structure of the network of interbank loans that is obtained by this procedure. In order to maintain at least some readability of the picture for each bank only its interbank-loan with the highest volume is shown:

6.4 The matrix of interbank participations

The matrix of participations - in percent of capital – of domestic banks in other domestic banks is directly reported from Austrian banks to the OeNB. In order to capture the full structure of participations we have to consider to different cases:

- Direct participations of domestic banks in domestic banks
- Indirect participations of domestic banks in domestic banks, where at least one institution in the chain of direct participations, on which this indirect bank to bank participation relies, is not a bank. Typically this is the case if one or more banks have participations in a holding-company, which itself hold participations in one or more banks.

As of end 2005 SRM fully captures the first, but not the latter type of interbank participation. However, indirect participations will be included as well in the near future.

6.5 Calculation of Capital

The calculation or definition of capital, respectively, is crucial for SRM as it defines the risk bearing capacity of bank that is how much capital is available to a bank in

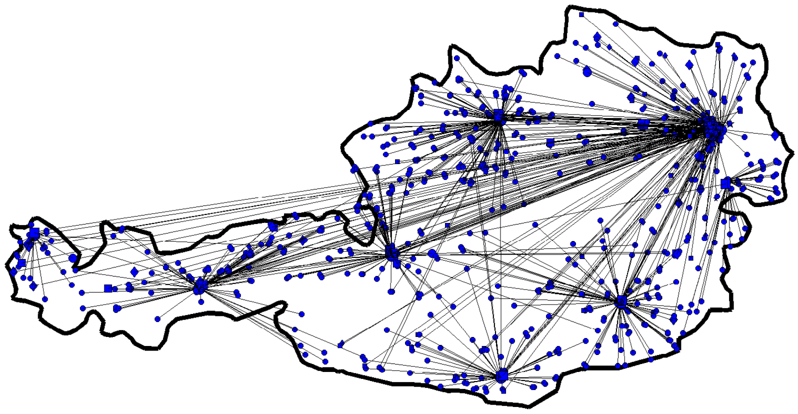

Figure 6.1. *Regional Structure of Austrian Interbank Loans*

order to cover losses from market, credit or contagious risk. As these resources to capture potential losses may not only include capital in the classical sense like tier1- or tier2- capital, but also other this like provisions or profits, the notion capital was set under quotes in the beginning of this paragraph. However, in the terminology of SRM we mean by capital all resources to capture potential losses. If the losses exceed capital, the bank defaults. SRM provides several options for the definition of capital ranging from a very strict to a rather broad definition:

- Tier 1 capital only is considered.
- Regulatory capital as defined by the Austrian banking act is considered
- Regulatory capital as defined by the Austrian banking act plus actual general and specific provisions for loans is considered

Depending on the choice for the calculation of capital results regarding the number of defaults occurring in a given simulation can vary quite substantially. However the choice of the capital basis to capture potential losses will be part of an ongoing process during which the practical use of SRM shall be explored in detail.

Chapter 7

Implementation

7.1 SRM Software Requirements

7.1.1 Scope and Context

To make SRM operational, it is implemented such that it can be accessed via an interface called from the analyst's desk. The interface is a Java client application which gives users the possibility to run certain predefined simulations (including a variety of regular stress tests) as well as to parameterize individual simulations. In either case output is written to Microsoft Excel files for further analysis, which are sent as an e-mail attachment to the analysts desk by SRM after a simulation request has been finished.

7.1.2 SRM Input

The main sources of data used by SRM are described in detail in chapter 6. These are bank balance sheet and supervisory data from the monthly reports to Oesterreichische Nationalbank (OeNB) and the database of the OeNB major loans register. In addition default frequency data in certain industry groups from the Austrian rating agency, financial market price data from Bloomberg, and macroeconomic time series from OeNB, the OECD and the IMF International Financial Statistics are used. These are varied data sources with quite different technical and structural properties. SRM provides means to read and aggregate data from multiple data sources with different data structures as well as varying degrees of frequency.

7.1.3 SRM Output

SRM output consists of descriptive system (risk) statistics of individual simulation runs on individual bank and various aggregates. Within SRM four main risk concepts are used:

- Fundamental and contagious defaults
- Pd distribution according to rating classes
- Aggregate loss distributions

- Quantification of resources that might have to be mobilized by a lender of last resort

Accordingly SRM writes output to Excel files to grant users the possibility to further investigate the statistics along these risk concepts.

7.1.4 The different uses of SRM

SRM is mainly used in three different ways:

- regular quarterly risk assessments,
- infrequent ad hoc simulations, and
- infrequent in depth analysis.

As the three scenarios differ in complexity, they demand increasing familiarity with SRM, which is reflected in an intuitive three layer approach. The first use corresponds to the regular system stability analysis conducted at the Financial Analysis Division of OeNB at the beginning of each quarter as soon as all new data are available. It consists of a simulation and a certain set of predefined stress tests. As this is regular analysis, SRM provides predefined model configurations as well as a set of predefined stress tests, to perform the task with a couple of mouse clicks. The concept of ad hoc simulations corresponds to irregular analyst needs, which are centred on a topical financial stability problem. It is catered for by SRMs advanced model and scenario options, which can be set from the interface where users can choose from a set of different SRM models (for instance varying credit- and market risk models) as well as define their main parameters. Additionally the SRM user interface provides the possibility to customize individual stress tests according to the demand of the ad hoc analysis. Finally the third complexity level, which provides means to fine tune SRM models and for instance evaluate model choices themselves, moves SRM configurability beyond the constraints of the interface. It relies on numerous configuration files, which contain additional parameters in comparison to the standard interface. This kind of configuration, however, is reserved for expert SRM users.

7.2 SRM Software Architecture

7.2.1 SRM Program Structure

SRM is implemented based on 2-tier architectures [see fig.]. A Java client application serves as graphical user interface and compiled, object-oriented Matlab code of SRM models and internal data structures as main application. External data is based on multiple sources and ranges from the local LDAP server for user authorization to Bloomberg market data that is read from flat files. SRM output is written to Excel files for further analysis.

7.2.2 SRM Presentation Layer

The SRM presentation layer is implemented as a client application, which grants the user the possibility to run certain predefined simulations with SRM or parameterize individual simulations. The chosen parameters are stored at database level and

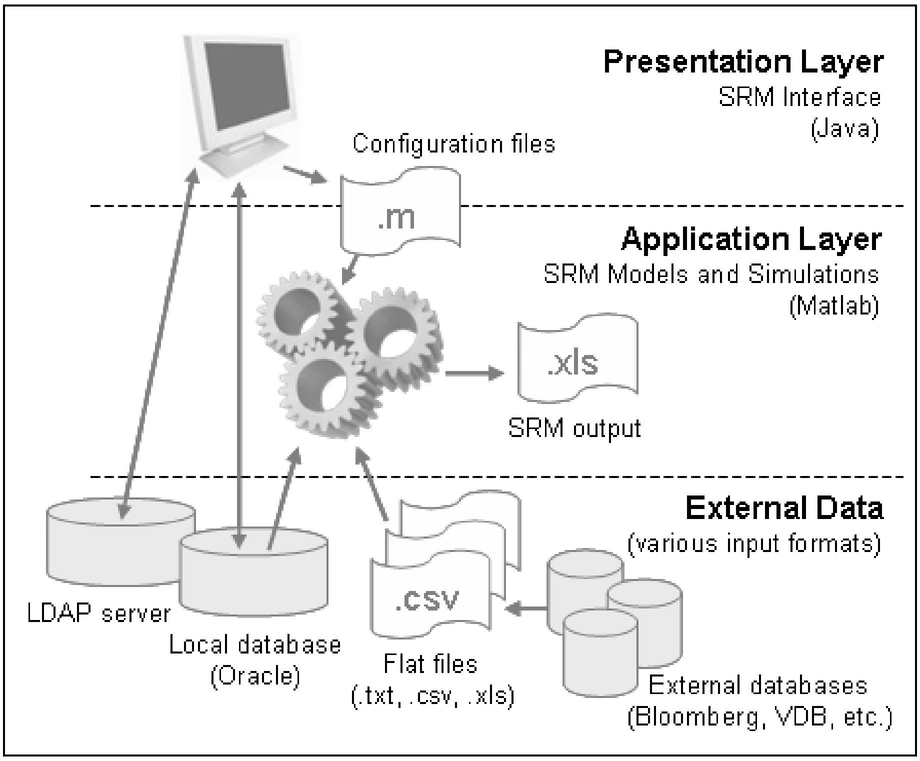

Figure 7.1. *SRM Software Architecture*

written to M-Files, which in turn are interpreted at run-time by the SRM application layer. The user interface is implemented with Java 1.4.2, data is held persistent on the same Oracle 10g database that contains some of SRMs input data. For usability reasons the authorization of users relies on their profiles of the OeNB LDAP server, which also holds their email addresses for notification purposes after a full SRM simulation.

7.2.3 SRM Application Layer

The SRM application layer is implemented with Matlab, version 14.3. Matlab is a matrix-based programming language for technical computing with data structure, function, and object-oriented programming features . The application layer of SRM has been implemented as a set of object-oriented M-files (Matlabs functions and applications) in Matlabs working environment, which includes tools for developing, managing, and debugging M-files. Additionally to Matlabs standard function library, SRM makes use of so called toolboxes. In particular, the financial, optimization, statistics toolboxes as well as the open source econometrics toolbox (see James [1999] were used. The application layer also contains the internal data structure of SRM. There are routines for data input and data output as well as objects that store the data during runtime. The source code can be executed via Matlabs standard user interface. However, in its end-user implementation, the code is compiled as C Code, and called via SRMs Interface. In either implementation output is written as Excel files for further analysis.

7.2.4 SRM Implementation

In more technical terms, SRMs application layer is comprised of four Matlab classes:

- the DataClass,
- the SimulationClass,
- the ModelClass,
- and the ScenarioClass.

The DataClass contains the banking data as well as time series data for macro, market and credit risk factors. Additionally general information about one simulation is stored in the DataClass, for instance the date for which a simulation is executed or information on the directory, where output should be stored. Each simulation in turn consists of n different runs of SRM. One run usually signifies a default SRM simulation, other runs stress test the system concerning the impact of different (groups of) risk factors. The information about individual runs is stored in the SimulationClass, which also handles the sequence of all runs of a simulation as well as the export of results. However, all the runs of one simulation share a certain set of SRM models (market risk model, credit risk model, etc.), which are stored as function calls in and according to parameters of the ModelClass. SRM is implemented in an object-oriented manner, therefore the possibility to easily add additional models is guaranteed, as long as they comply to the implemented interfaces. Finally there is the ScenarioClass, which contains the market and credit risk scenarios for each individual run based on the data from the DataClass, the choice of stress parameters from the SimulationClass and the models from the ModelClass.

7.3 SRM Data Design

7.3.1 SRM Input Data Structure

For an extensive description of the input data sources and their structures refer to the previous chapter. Within the SRM data is handled by the DataClass, which provides all the necessary functions to read and structure data from the various external sources.

7.3.2 SRM Data Class

The SRM DataClass provides two fundamental functionalities to SRM. As its name suggests, it contains all of SRMs data in a structured manner, which as the object-oriented paradigm would suggest is also read and aggregated within the DataClass. A major role is therefore the provision of data to other classes of SRMs application layer. Secondly the DataClass contains all handling of general SRM settings, whether it is the information on the runtime environment (which risk factors are includes, what data sources are to be used, where it is read from, and where to store SRMs output), in or out functions or the progress log.

Object variables The SRM DataClass stores two types of data, bank specific data and market data. Bank specific data is organized in four object variables of the DataClass as follows:

- Banks is an array of banks, with information regarding each individuals code, name, sector, etc.
- Positions is an array of all reported positions including information such as their code, name, currency, etc.
- Firms is an array of those firms and individuals for which banks report large exposures
- BankValues is the object variable that stores the actual values whereas the other three just hold the information about the data Besides this bank specific data, the DataClass stores market data, too.

  This data is organized in the following three object variables:

- Timeseries is an array of time series, containing the information about each time series in SRM
- Dates is a vector of dates
- MarketValues is the object that stores the actual values for each time series and date specified above

Additionally the DataClass contains an object variable Options, which in turn contains general SRM parameter such as the current version, and runtime related variables. Other Option parameters are concerned with the description of the aforementioned bank specific and time series data.

7.3.3 SRM Output Data Structure

SRM output consists of descriptive system (risk) statistics of individual simulation runs on individual bank and various aggregate levels. SRM writes its output to Excel files to grant users the possibility to further investigate these statistics. For each run there is a separate Excel file, which contains one or more sheets of the following six categories:

- Parameter sheets
- System (risk) statistics sheets
- Rating sheets
- Lender of last resort sheet
- Scenario sheet

The parameter sheets contain information on the parameters of the simulation run. They are provided in three levels of granularity according to the three identified usage scenarios. They therefore range from one reduced sheet, which contains the name of the run, information on potentially stressed risk factors and the choice of models, to the most detailed sheet, which contains all of the almost 200 SRM parameters. The system (risk) statistics sheets contain the main SRM output, probabilities of default for fundamental and contagious default over the course of the next quarter and projected values for a year. It also includes aggregated loss distributions for market-, credit- and contagion risk losses as VaRs for different percentiles. Each statistics sheet is provided in absolute values and on another sheet relative to the current capital definition (which can also be parameterized in SRM). Values are calculated for the aggregate system and on an individual bank level, as well as on any level that banks can be grouped (size, sector, etc.), which results in two statistics sheets (absolute and relative) per level in the SRM output file. In the so called rating sheets, SRM provides two tables of OeNB rating classes (one more detailed than the other), to which individual banks are mapped according to their probability of default during the particular simulation run. The lender of last resort sheet provides a quantification of resources that might have to be mobilized by a lender of last resort, structured by total-, market-, credit-, and contagion risk losses. And finally the scenario sheet, which provides an overview of the SRM scenarios according to the number of defaults of individual banks, that occurred within each individual scenario of the current simulation run.

7.4 SRM Functional Design

7.4.1 SRM Functional Sequence

SRM is composed of three main functional blocks [see fig.]:

- the initialization,
- the simulation, and
- the calculation of the results.

During initialization all four SRM object classes are initialized, their parameters are set according to the values in the configuration files. The next steps include the initialization of SRMs network model, the initialization of the scenarios (i.e. the market and credit risk models) and the simulation runs. Initialization is the one functional block that all runs of one SRM simulation share, as each individual run uses the same network model as well as the same market and credit risk models. During the simulation phase, each individual run (stressed or unstressed) undergoes the same three steps:

- the calculation of market risk
- the calculation of credit risk, and
- the network clearing.

The former two consist again each of two steps, the generation of market and credit risk scenarios (on the ScenarioClass side) for which losses are calculated (on the ModelClass side). The final simulation step for each run is the clearing of the network, which includes the calculation of losses due to contagion. Finally the results are calculated for each individual run of the simulation and exported to Excel files for further analysis (which is handled by the SimulationClass). The separation of simulation and result calculation serves the purpose of an easier extension of either end, but particularly the calculation and export of results, as quite a lot of selection and aggregation of the generated statistics has to be performed.

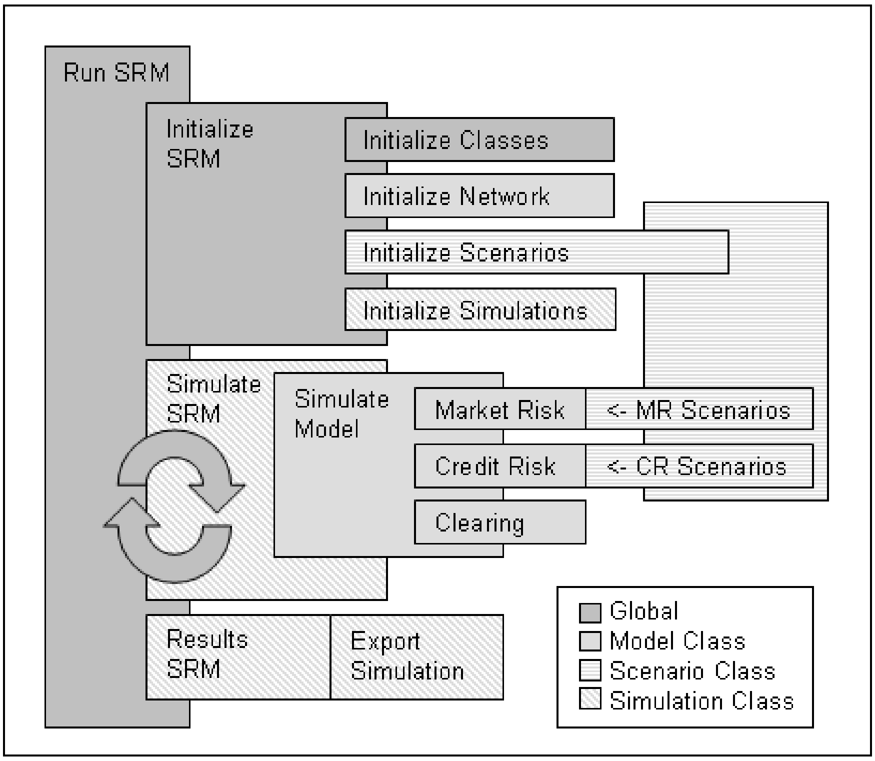

Figure 7.2. *SRM Sequence Diagram*

7.4.2 SRM Simulation Class

The SRM SimulationClass provides two functionalities to SRM. As its name suggests, it contains the management of individual simulation runs, in case of one run being a stress test it also includes the management of the stressed risk factor(s) and the kind of stress that has to be applied. Secondly the SimulationClass handles the calculation and export of the results of each individual simulation run.

Object variables

Other than the object variable Options, which contains general saving and export parameter, the SRM SimulationClass stores the structure and values of stressed risk factors (macro and market time series as well as default rates for different sectors) alongside general information on how to handle each simulation run.

Functions

7.4.3 SRM Model Class

As its name suggests the ModelClass contains the SRM models. That is the implementation of these models as functions of the ModelClass and the information on which of the available models are used for the current simulation. The models in question include:

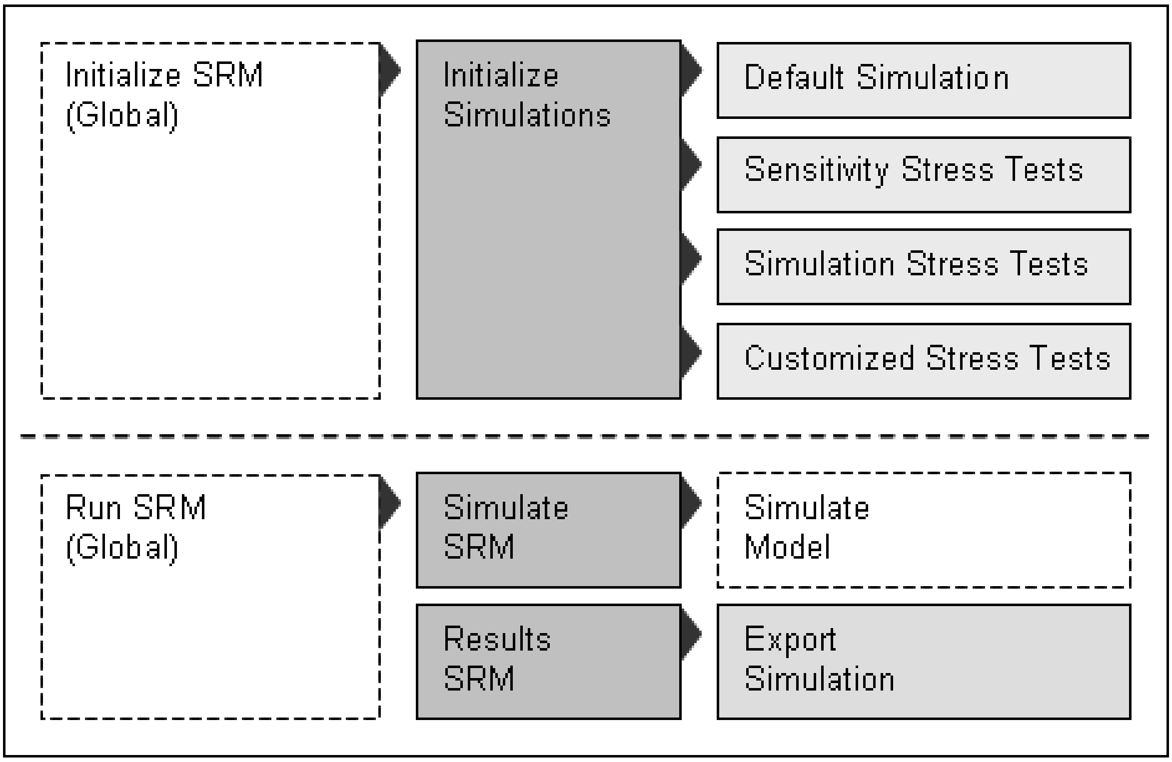

Figure 7.3. *SRM Simulation Class Functions*

- the network model
- the market risk model, either one of the following:
  - historical simulation
  - multivariate normal model
  - copula model
- the credit risk model, either one of the following:
  - expected loss calculation
  - macro model monte carlo
  - macro model credit risk plus
- and the network clearing model

Object variables

The SRM ModelClass stores the information which model to use in an object variable called Options, which also contains the parameterization of each of the models in use. Additionally it is the ModelClass that stores the (banks by scenario) matrices for

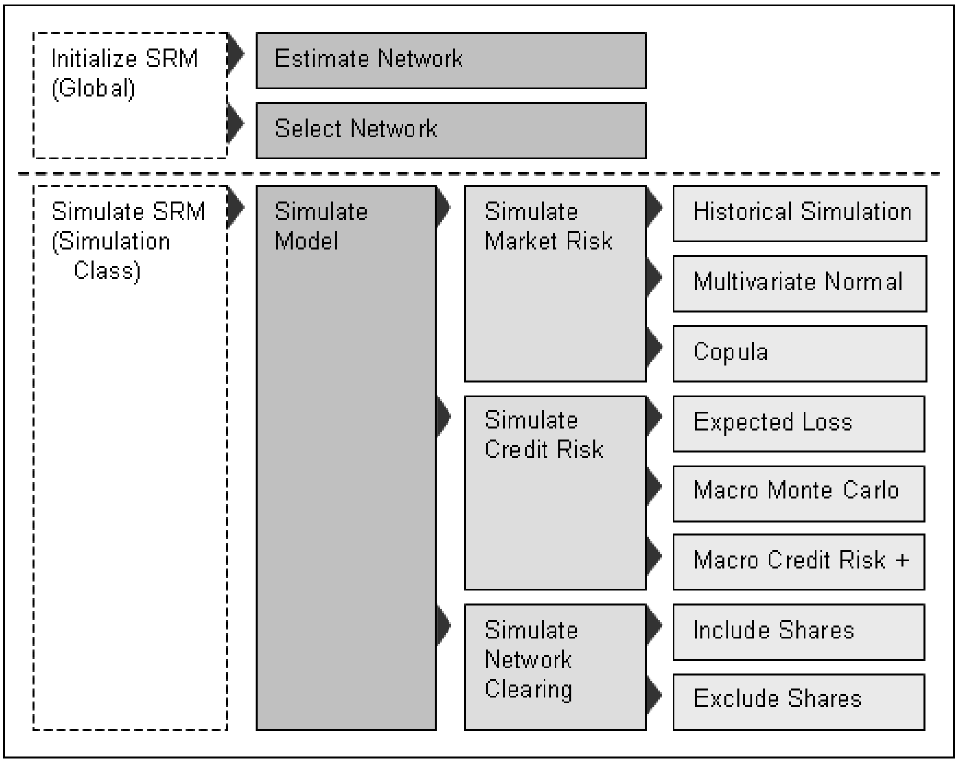

Figure 7.4. *SRM Model Class Functions*

- the network exposures,
- market- and credit risk losses,
- contagious losses, and
- defaults.

This also indicates one ModelClass object exists for each individual run of a simulation as opposed to Data- and SimulationClass, for which one object exists per simulation.

Functions

7.4.4 SRM Scenario Class

The ScenarioClass contains the scenarios for each run of a SRM simulation. That is the implementation of the scenario generation as functions of the ScenarioClass and the information on which of the available functions to generate scenarios are used for the current simulation. The methods in question include:

- the market risk scenarios, either one of the following:
  - historical scenarios

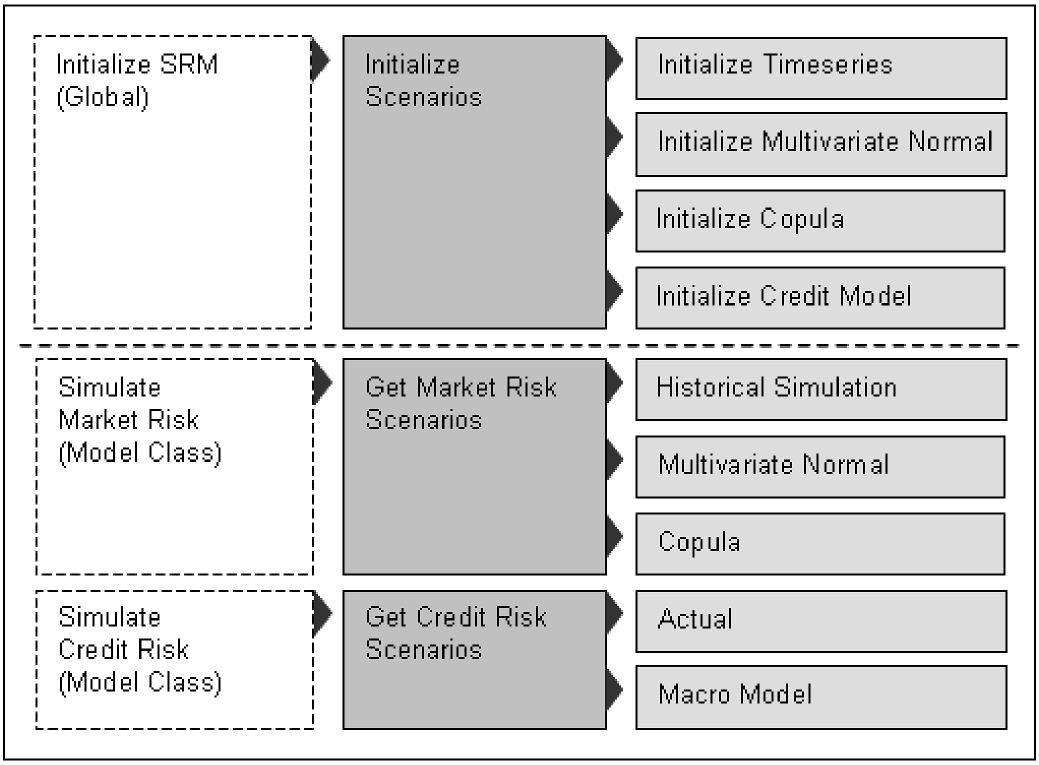

Figure 7.5. *SRM Scenario Class Functions*

  - multivariate normal distributed scenarios
  - scenarios based on a t-copula

- the credit risk scenarios, either one of the following:
  - scenarios according to current default rates
  - scenarios based on the simulated macroeconomic risk factors

Object variables

The SRM ScenarioClass stores the information which methods to use in an object variable called Options, which also contains the parameterization of each method in use. Additionally the ScenarioClass stores the generated macroeconomic risk factor scenarios, as they serve as conditional scenarios for the market risk factors. However market risk scenarios are by default not stored for performance reasons, but can be saved to hard drive if necessary.

Functions

7.5 SRM Interface Design

7.5.1 SRM Interface

The interface is a Java client application which aims at high usability for the two main SRM usage scenarios:

- regular quarterly risk assessments and
- infrequent ad hoc simulations.

The interface therefore gives users the possibility to run certain predefined simulations as well as to parameterize individual simulations. The former corresponds to the regular system stability analysis conducted at the Financial Analysis Division of OeNB at the beginning of each quarter as soon as all new data are available. It consists of a default simulation and a certain set of predefined stress tests. As this is regular analysis, SRM provides default model configurations as well as a set of predefined stress tests, which can be selected and executed from the interface main menu. The later corresponds to infrequent simulations, which are centred on a topical financial stability problem. It is catered for by SRMs advanced model and scenario options, which can be set from the interface sub menus, where users can choose from a set of different SRM models as well as define their main parameters. Additionally the SRM user interface provides the possibility to customize individual stress tests according to the demand of the ad hoc analysis. In either case output is written to Microsoft Excel files for further analysis, which are sent as an e-mail attachment to the analysts desk by SRM after a simulation request has been finished.

7.5.2 Interface Sequence

7.5.3 Main Menu

SRMs main menu consists of four functional groups [see fig.]:

- data configuration,
- bank options,
- simulation types, and
- advanced options.

Data configuration allows the user to set the point in time for which a simulation is conducted, additionally certain database options can be set. Bank options let the user define which institutes to include in the simulation and simulation types provide the aforementioned default simulation as well as the predefined stress tests. These consist of a sensitivity analysis for each of the included risk factors, and more complex regular stress simulations such as a GDP shock or a hike in EUR interest rates. The choice of stress test customization leads to the menu simulation configuration, where individual stress tests can be defined. Finally the advanced options, which also lead to further interface menus, namely the advanced model options for model selection and parameterization and the advanced scenario options for scenario generation and parameterization.

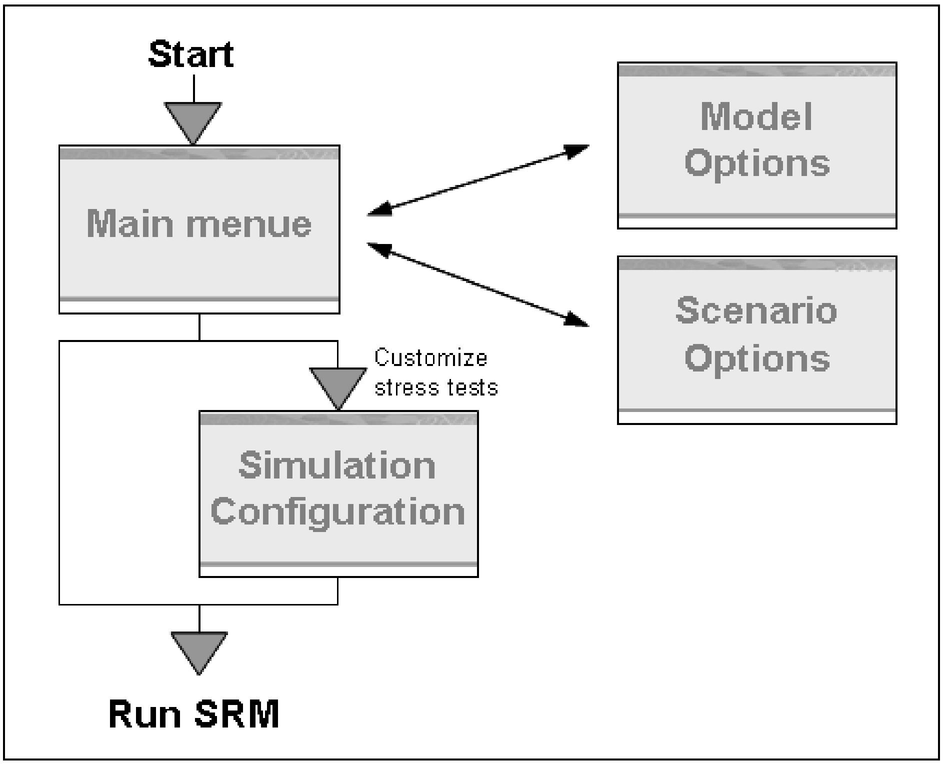

Figure 7.6. *SRM Scenario Class Functions*

Parameter description

The following SRM parameter can be set from the main menu interface (see tab., for a complete listing of parameters including their data types, default values, and a short description):

Insert Table here

7.5.4 Advanced Model Options

SRMs advanced model options menu consists of three functional groups [see fig.]:

- market model options.
- credit model options, and
- clearing model options.

For each of the three model choices are provided, including the calibration of the main parameters according to the selected model.

Parameter description

The following SRM parameter can be set from the advanced model options interface (see tab., for a complete listing of parameters including their data types, default values, and a short description):

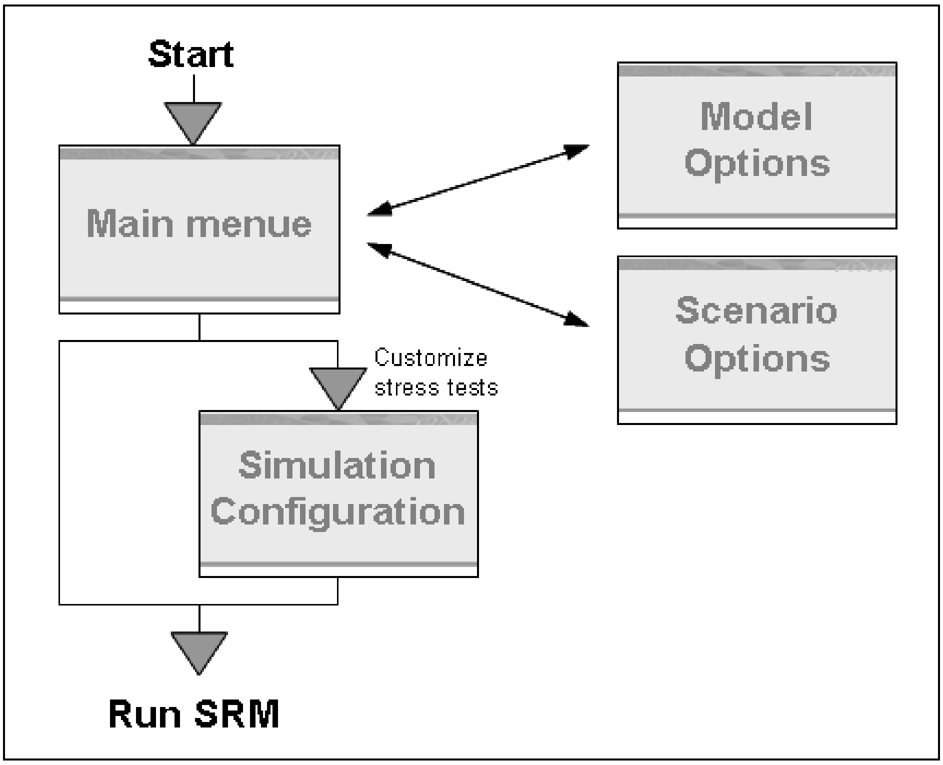

Figure 7.7. *SRM Interface Main Menu*

insert table here

7.5.5 Advanced Scenario Options

SRMs advanced scenario menu consists of four functional groups [see fig.]:

- market scenario options
- macro model copula parameter options,
- credit scenario options, and
- credit model parameter options.

Market scenario options allow the user to set certain parameter regarding the treatment of time series when estimating market risk scenarios, whereas macro model copula parameter options provide the possibility to influence the copula estimation (for example the kind of error distribution used). Credit scenario options provide the choice of different models for credit risk scenario generation, and credit model parameter allow users to set the credit model parameter, given credit generation is set to the option "macro model".

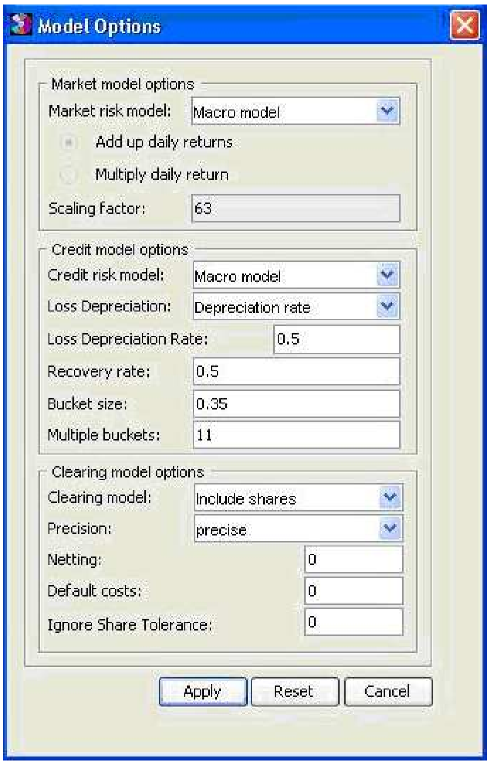

Figure 7.8. *SRM Interface Advanced Model Options*

Parameter description

The following SRM parameter can be set from the advanced scenario options interface (see tab., for a complete listing of parameters including their data types, default values, and a short description):

7.5.6 Simulation Configuration

Although SRMs simulation configuration menu equally consists of three functional groups [see fig.], they differ in regard to their structural properties:

- simulation overview manages the individual runs of a simulation
- and a little misleading, simulation contains the parameters of each run, and is split into:
  - parameter, which contain the general parameter of an individual run and
  - riskfactors, which provides the possibility to stress each risk factor of one run

The simulation overview serves as the console to manage the customized runs. Parameters can not be changed at that level, as it is simply a display of the currently defined runs. However, one can add and delete individual runs, as well as reset any

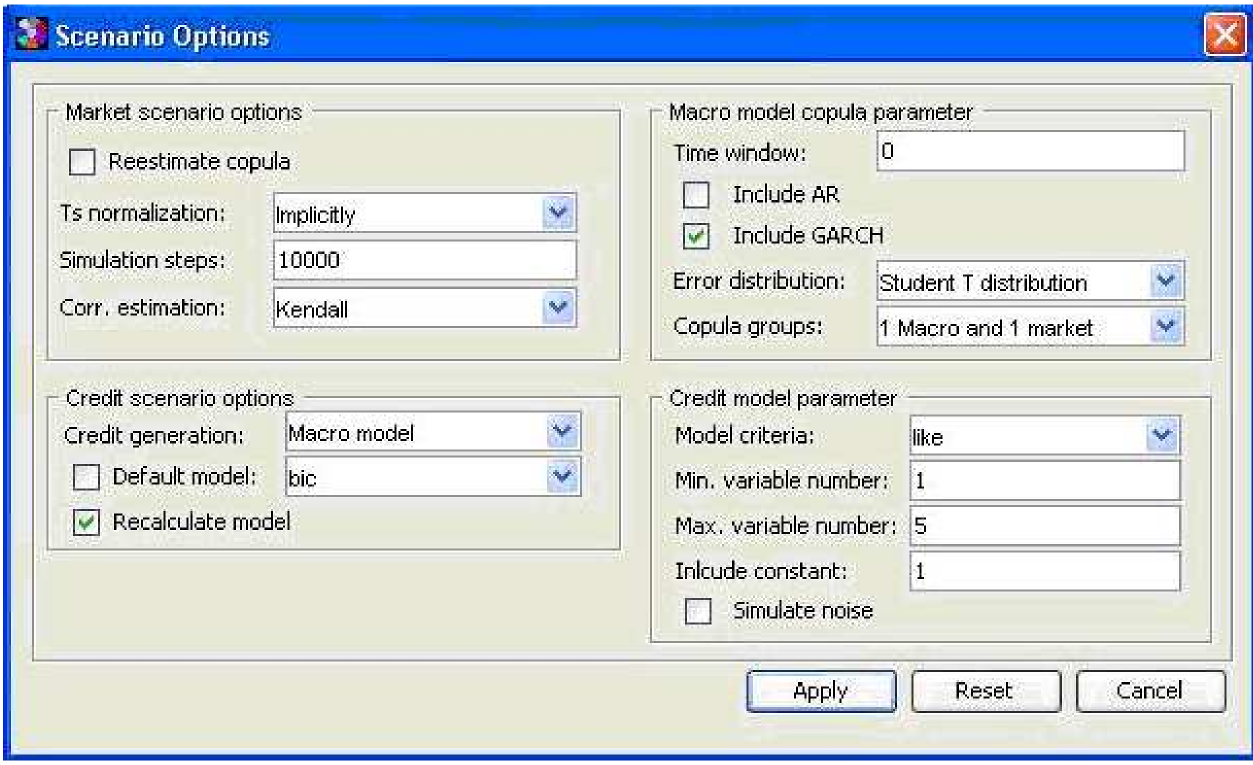

Figure 7.9. *SRM Interface Advanced Scenario Options*

run to general default values. The simulation overview also enables users to select one run for display, to actually define the parameter.

Parameter description

The following SRM parameter can be set from the simulation configuration interface (see tab., for a complete listing of parameters including their data types, default values, and a short description):

Insert Table here

Additionally stress factors can be defined for three groups of risk factors:

- macro factors (Macro),
- market factors (Market), and
- probabilities of default (Credit Pds).

Each group of risk factors can be accessed by a separate panel in the lower third of the interface. Risk factors that are listed there and can be stressed in three different ways:

- set the value of a risk factor to a certain level (set),
- in- or decrease the current value of a risk factor by a certain value (abs), or

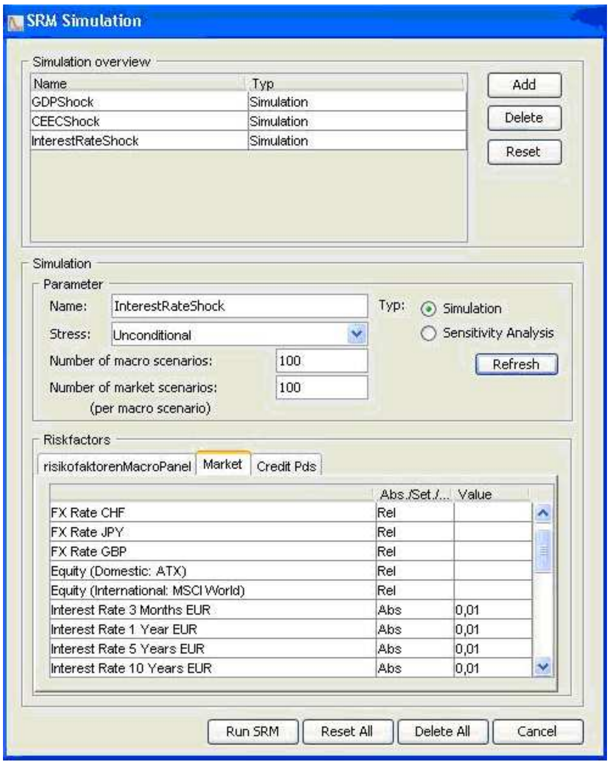

Figure 7.10. *SRM Interface Simulation Configuration*

- a relative in- or decrease of the current value

However, as stress for macro factors is defined as a deviation from an estimated baseline, macro risk factors can currently not be set to a certain value.

7.6 SRM Restrictions, Limitations and Constraints

7.6.1 Constraints of the Interface

Adding new parameter
Adding new models
Changing risk factors
Risk factor levels

7.6.2 Constraints within the Application

Default credit model
Stressing macro risk factors

7.6.3 Constraints of the Output

Output parameter

7.7 SRM Performance

Appendix A

Appendix

A.1 Modelling Multivariate Distributions of Risk Factors

Our approach to modeling multivariate distribution of risk factors is based on the Sklar's Theorem (Sklar [1959]).

Sklar's theorem: Let F be a multivariate distribution function with marginal cumulative distribution functions F_i for $i = 1, \ldots, n$. Then there exists a function $C : [0,1]^n \to \mathbb{R}$ satisfying

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{A.1}$$

C is called the copula. If the marginal distributions F_i are continuous, then C is unique. When C is a copula and F_i are arbitrary distribution functions, then the function F defined by equation (A.1) is the multivariate function of a multivariate distribution with marginals F_i.

Sklar's Theorem give us the possibility to model the multivariate distribution function in two steps, namely modeling of marginal distributions (Section A.1.1) and modeling the copula (Section A.1.3).

A.1.1 Modelling and estimating the marginal distributions

Denote the given time series as S_t, $t = 1, \ldots, n$. We model the log-returns of the risk factors

$$r_t = \ln(S_{t+1}) - \ln(S_t) \quad t = 1, \ldots, n.$$

We chose not to model AR effects in the market time series, because no-arbitrage arguments deny the significance of autocorrelations effects for traded prices.

Aggregating higher frequency models to 60 day models

We concentrate on a time horizon of 60 trading days. Market data are available at a higher frequency, usually daily or even intra-day. This opens the possibility to model the distributions of log-returns over shorter periods (1, 5, 10, 20, 30, or 60 days), and then aggregate these distributions in order to arrive at forecasts of 60 days returns. In this way one can possibly exploit the availability of higher frequency data in order to get more reliable estimates. On the other hand, estimation and or modelling errors might be magnified by the aggregation.

When we confine ourselves to the standard deviation of the return distribution, a simple aggregation method is given by multiplying by the square root of time. This works correctly if we have i.i.d. returns. Volatility clustering observed in the markets implies that returns are not i.i.d. This finding questions the appropriateness of the square root of time-method.

Let us denote an m-period return during time t and $t+m$ by ${}_m r_t$. (We abbreviate the 1-period return ${}_1 r_t$ between times t and $t+1$ by r_t.) As we consider log-returns, we have ${}_m r_t = \sum_{i=0}^{m-1} r_{t+i}$. When the returns r_t are discrete, the probability that ${}_m r_t$ is equal to y is the probability that $\sum_{i=0}^{m-1} r_{t+i}$ is equal to y. In this case one has to sum the probabilities of all possible paths of $\{r_{t+i}\}_{i=0}^{m-1}$ which sum up to y. In analogy, for continuous one-period returns the density function of ${}_m r_t$ is

$$f_{{}_m r_t}(y) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{r_{t+m-1}} \left(y - \sum_{i=1}^{m-1} x_i \right) \prod_{i=1}^{m-1} f_{r_{t+i-1}}(x_i) dx_1 ... dx_{m-1}, \qquad (A.2)$$

where f_{r_t} denotes the density function of the one-period returns r_t, conditional on the previous realizations of r_t.

In general, this multi-period density function can be evaluated only numerically. Therefore, we use a Monte Carlo simulation in order to approximate the aggregated distribution function of the density (A.2). We simulate 10.000 paths of m steps by drawing for each step from the distribution given by the density f_{r_t}. Each path yields a value after step m, which is a draw from the aggregated distribution. Simulating enough paths gives a sufficient approximation of the aggregated distribution.

GARCH

GARCH processes are a popular tool for the description of financial time series because they are known to describe volatility clustering. Aggregated GARCH processes can also capture the heavy tails even if the one-period distribution is normal.

Definition *The sequence $\{r_t, t \in \mathbb{Z}\}$ is defined to be generated by a* strong GARCH(1,1) process *if*

$$\begin{aligned} \epsilon_t &:= r_t/\sigma_t \sim D(0,1) \qquad i.i.d., \\ \sigma_t^2 &= a + b r_{t-1}^2 + c\sigma_{t-1}^2, \end{aligned} \qquad (A.3)$$

where $D(0,1)$ specifies a distribution of errors with mean zero and unit variance.

The class of strong GARCH processes is somewhat restricted because it assumes that the errors are identically distributed and independent. (Strong, semi-strong, and weak GARCH processes are defined e.g. in Drost and Nijman [1993].) Still, in the sequel, we only consider strong GARCH processes

$$r_t = \sigma_t \epsilon_t, \qquad (A.4)$$

where ϵ are iid with some distribution, with zero mean and unit variance and volatility modelled as

$$\sigma_t^2 = a\sigma_{t-1}^2 + br_{t-1}^2 + c. \tag{A.5}$$

In case, where the GARCH effects are not included in the model, the σ_t is set to a constant for all t.

The GARCH parameters can be estimated with the quasi maximum likelihood method (see e.g. Berkes et al. [2003], McNeil and Frey [2000]). This method introduces two additional assumptions. The first is that ϵ is normally distributed (which actually it is not). The second assumption is $a + b < 1$ to guarantee stationarity of the time series (see i.e.Drost and Nijman [1993]). With these two assumptions the estimation proceeds as the usual maximum log likelihood method. Our GARCH(1,1) model depends on the parameters a, b, c, σ_0^2 defined by

$$\ln(lf) = \frac{1}{2}\sum_{i=1}^{n}\left(\ln(\sigma_{i-1}^2) + \frac{r_i^2}{\sigma_{i-1}}\right) + \frac{n}{2}\ln(2\pi), \tag{A.6}$$

where σ_0^2 is an exogenous starting value and the σ_t are defined by equation A.5. Although the quasi maximum likelihood procedure relies on the normality assumption which is known to be violated, the practical applicability of estimation results is ensured by various asymptotic results, see i.e. Bollerslev and Wooldridge [1992].

Distribution of errors

Another ingredient of our marginal models is the distribution of errors ϵ in equation (A.4).

For the error distributions we admit three possible models.

- **norm** - uses normal errors. To estimate the parameters we take the usual estimators of mean and variance,

$$\mu = \frac{1}{n}\sum_{t=1}^{n}\epsilon_t$$
$$\sigma^2 = \frac{1}{n-1}\sum_{t=1}^{n}\left(\epsilon_t - \mu\right)^2, \tag{A.7}$$

- **t** - uses the Student distribution.This allows for modelling heavier tails. Actually, we model the errors as $\epsilon \sim a + bt_\nu$, where t_ν represents the Student distribution with ν degrees of freedom. Since the Student distribution has zero mean, variance equal to $\frac{\nu}{\nu-2}$, and kurtosis equal to $\frac{6}{\nu-4} + 3$, we can estimate a, b, ν in the following way. In terms of these parameters mean, variance, and kurtosis of ϵ are given by

$$\mu_1(\epsilon) = a + b\mu_1(t_\nu) = a, \tag{A.8}$$
$$\mu_2(\epsilon) = b^2\mu_2(t_\nu) = \left(b\frac{\nu}{\nu-2}\right)^2, \tag{A.9}$$
$$\frac{\mu_4(\epsilon)}{\mu_2^2(\epsilon)} = \frac{b^4\mu_4(\epsilon)}{(b^2\mu_2(\epsilon))^2} = \frac{\mu_4(\epsilon)}{\mu_2^2(\epsilon)} = \frac{6}{\nu-4} + 3, \tag{A.10}$$

where μ_i is the i-th central moment. From equations (A.8, A.10) we directly get the parameters a and ν. From ν we can compute b by equation (A.9).

Alternatively, the parameters a, b and ν could be estimated by the maximum likelihood approach.

- **EVT** - uses a distribution with the 80% body taken from the empirical distribution (as in in historical simulation), and the 10% data in the left and right tails are used to estimate the distribution of tails (excess distribution) which is well approximated by the generalised Pareto distribution (see i.e. McNeil and Frey [2000], Pickands [1975])

$$G_{\xi\sigma}(y) = 1 - \left(1 + \frac{\xi}{\sigma} y\right)^{-1/\xi}. \tag{A.11}$$

We estimate σ and ξ of the generalised Pareto distribution with maximum likelihood method, using as objective function the log-likelihood

$$\ln(lf) = -m \ln(\sigma) - \left(1 + \frac{1}{\xi}\right) \sum_{i=1}^{m} \ln\left(1 + \frac{\xi}{\sigma} e_i\right), \tag{A.12}$$

where e represents the excess over the 10%-quantile of the empirical distribution.

A.1.2 Tests of the marginal distributions

In order to test the 60 days distribution forecasts produced by the various models, it is not enough to assess whether the means, variances, or some quantiles of the distributions were correctly predicted. (Back testing, for example, amounts to a test of a quantile of the predicted distributions.) For many applications the overall distributional properties are important, not just the means or variances. Therefore, based on DeRaaij and Raunig [2002], we test for the adequacy of the density forecasts of the entire distribution.

Consider a time series of returns r_t $(t = 1, \ldots, n)$ generated from some true conditional densities $f_t(.)$ $(t = 1, \ldots, n)$. Now some model produces a series of 60 days conditional density forecasts $p_t(.)$ $(t = 1, \ldots, n)$. The task is to evaluate whether the true conditional densities $f_t(.)$ agree with the predicted conditional densities $p_t(.)$. Applying the Rosenblatt transformation (see Rosenblatt [1952]) to the observed returns r_t,

$$r_t \mapsto z_t := \int_{-\infty}^{r_t} p_t(u)du \tag{A.13}$$

we get a transformed series z_t which should be i.i.d. U(0,1) if the predicted conditional densities $p_t(.)$ agree with the true conditional densities $f_t(.)$. Applying the inverse of the normal distribution function

$$z_t \mapsto n_t := \Phi^{-1}(z_t), \tag{A.14}$$

produces a series n_t which is standard normally i.i.d. if the original returns r_t are distributed according to the predicted densities p_t (see Berkowitz [2001]).

We then perform a Kolmogorov-Smirnov (KS) test for the hypothesis that the n_t are sampled from a standard normal distribution. The KS-test compares the sample values with a standard normal distribution. The null hypothesis for the Kolmogorov-Smirnov test is that the sample has a standard normal distribution. The alternative hypothesis that the sample does not have that distribution. For each potential value x, the Kolmogorov-Smirnov test compares the proportion of values less than x with the expected number predicted by the standard normal distribution. The test statistic is the maximum difference over all x values. The Kolmogorov-Smirnov test is moderately sensitive to the whole distribution including the first and second moment, however fit of the tails is not tested very sensitively. A model is accepted if the p-value is higher than 5%.

In order to test additionally whether the *conditional* variance of the n_t is constant and equal to one, DeRaaij and Raunig [2002] consider the regressions

$$n_t = \beta_0 + \beta_1 n_{t-1} + u_t \tag{A.15}$$

$$n_t^2 = \gamma_0 + \gamma_1 n_{t-1}^2 + v_t \tag{A.16}$$

where u_t and v_t are non-autocorrelated with zero expectation conditional on their own past values. In case the n_t have zero mean and are uncorrelated we have $\beta_0 = 0$ and $\beta_1 = 0$. In case the n_t have constant conditional unit variance we have $\gamma_0 = 1$ and $\gamma_1 = 0$. To test whether these restrictions are satisfied, DeRaaij and Raunig [2002] propose a joint Wald test of the four equalities $\beta_0 = 0$, $\beta_1 = 0$, $\gamma_0 = 1$, and $\gamma_1 = 0$. Additionally, they use the Jarque-Bera (JB) test to see whether the n_t have skewness zero and kurtosis equal to three. The Jarque-Bera test without the Wald test would not be very powerful since it does not test for mean and variance. The JB-test evaluates the hypothesis that the sample has a normal distribution with unspecified mean and variance, against the alternative that the sample does not have a normal distribution. The test focuses on the sample skewness and kurtosis. For a normal distribution, the sample skewness should be near 0 and the sample kurtosis should be near 3. The Jarque-Bera test determines whether the sample skewness and kurtosis differ significantly from their expected values, as measured by a chi-square statistic.

To sum up, we perform the following Test 2. A model is accepted if the p-value of the Jarque-Bera test is higher than 5% *and* the p-value of the joint Wald test for $\beta_0 = \beta_1 = \gamma_1 = 0$ and $\gamma_0 = 1$ is higher than 5%.

A.1.3 Modelling and estimating the copula

As already mentioned, our approach to modeling the multivariate density function is a two step procedure: First we model univariate time series. Then we model dependence separately by a grouped t-copula. This subsection will focus on the estimation of the copula. We will begin with short paragraph about the choice of the copula, and then give a procedure how to construct random variables distributed according to the grouped t-copula. Then we describe how we estimate the copula parameters - the correlation matrix and the degrees of freedom for the different groups. Finally we describe how to draw scenarios from the unconditional and conditional grouped t-copula.

Choice of copula

Some empirical observations suggest that the multivariate normal distribution is not optimal for modelling multivariate financial time series. For example, the multivariate normal exhibits zero tail dependence (see i.e. Fortin and Kuzmics [2002]), while real data often exhibit non-zero tail dependence. To allow for possible tail dependence one of several possible choices is the Student copula. However, if more than two risk factors are linked by a Student copula, all pairs exhibit the same tail dependence. This is clearly unrealistic. To allow for different tail dependence between different pairs of variables, we use the grouped t-copula (Daul et al. [2004]). In this project we did not perform any systematic tests to evaluate the appropriateness of other copulas.

Construction of the grouped t-copula

By Sklar's theorem the joint distribution is uniquely determined by the marginal distributions and the copula function. Our choice for the copula is the grouped t-copula. This choice is motivated—although not strictly implied— by the desire to have distributions with non-zero tail dependence, and possibly with different tail dependences between different risk factors. The usefulness of the t-copula was examined for example by Dias and Embrechts [2003] and Chen et al. [2004].

Daul et al. [2004] describe how the grouped t-copula arises. Let $\mathbf{Z}$ be normally distributed with means zero and correlation matrix ρ, and U a random variable uniformly distributed on (0,1), independent of $\mathbf{Z}$. Furthermore, let G_ν be the distribution function of $\sqrt{\nu/\chi^2_\nu}$, where ν is the number of degrees of freedom and χ^2_ν is the χ^2-distribution with ν degrees of freedom. Partition the set of variable indices $\{1, 2, \ldots, n\}$ into m sets of sizes $s_1, s_2, \ldots, s_m$. Let $R_k = G^{-1}_{\nu_k}(U)$ for $k = 1, \ldots, m$. If

$$\mathbf{Y} := (R_1 Z_1, \ldots, R_1 Z_{s_1}, R_2 Z_{s_1+1}, \ldots, R_2 Z_{s_1+s_2}, \ldots R_m Z_n)', \tag{A.17}$$

then the vector $(Y_1, Y_2, \ldots, Y_{s_1})'$ has an s_1-dimensional t-distribution with ν_1 degrees of freedom and, for $k = 2, \ldots, m$ the vectors $(Y_{s_1+\ldots+s_k+1}, \ldots, Y_{s_1+\ldots+s_{k+1}})'$ has an s_k-dimensional t-distribution with ν_k degrees of freedom. Finally, let F_k denote the distribution function of Y_k and let $H_1, \ldots, H_n$ be some arbitrary strictly increasing distribution functions. Then

$$\mathbf{X} := \left(H_1^{-1}(F_1(Y_1)), \ldots, H_n^{-1}(F_n(Y_n))\right)' \tag{A.18}$$

is a distribution with marginals $H_1, \ldots, H_n$ and the copula within each of the subsets $\{1, \ldots, s_1\}, \{s_1{+}1, \ldots, s_1{+}s_2\}, \ldots, \{s_1{+}s_2{+}\ldots{+}s_{m-1}{+}1, \ldots s_n\}$ being a $t{-}copula$ with ν_1, $\nu_2, \ldots, \nu_m$ degrees of freedom.

Two risk factors which are in the same group and which have linear correlation coefficient ρ_{12} have upper and lower tail dependence equal to

$$\lambda = 2 - 2t_{\nu+1}\left(\sqrt{\nu+1}\sqrt{1-\rho_{12}}\sqrt{1+\rho_{12}}\right),$$

where ν is the number of degrees of freedom of the group to which the two risk factors belong, and t_ν is the distribution function of the univariate Student t-distribution with ν degrees of freedom (see Embrechts *et al.* Embrechts et al. [2002]).

Estimation of grouped t-copula

With given groups we need to estimate two sets of parameters of the copula: the linear correlation matrix ρ and the number of degrees of freedom for each group.

Estimation of correlations

We estimate the correlation matrix from Kendall's tau τ. Assume we are given a series of n simultaneous observations (x_i, y_i) of the risk factors X, Y, where all

x_i differ from each other and all y_i differ from each other. In an ordering of the x-values, denote the rank of an observation x_i by $r(x_i)$. Similarly, in an ordering of the y-values, denote the rank of an observation y_j by $r(y_j)$. Now consider all possible pairings of the n observations. In total there are $n(n-1)/2$ pairings of observations. A pairing $((x_i, y_i), (x_j, y_j))$ is called concordant either if both $r(x_i) > r(x_j)$ and $r(y_i) > r(y_j)$ or if both $r(x_i) < r(x_j)$ and $r(y_i) < r(y_j)$. A pairing $((x_i, y_i), (x_j, y_j))$ is called discordant either if both $r(x_i) > r(x_j)$ and $r(y_i) < r(y_j)$ or if both $r(x_i) < r(x_j)$ and $r(y_i) > r(y_j)$. Denote the number of concordant pairs by C and the number of discordant pairs by D. Then Kendall's tau is defined by

$$\tau_\alpha(X, Y) := \frac{C - D}{n(n-1)/2}.$$

Kendall's tau is related to the linear correlation coefficient by

$$\rho_{X,Y} = \sin(\frac{\pi}{2}\tau_\alpha(X, Y)). \tag{A.19}$$

The main advantage of estimating linear correlation coefficients via Kendall's tau is the robustness of this method and its invariance under strictly increasing component-wise transformations T_1, T_2: $\tau_\alpha(X, Y) = \tau_\alpha(T_1(X), T_2(Y))$ (see i.e. Daul et al. [2004]). This in turn implies that the estimated correlation matrix does not depend on the number of degrees of freedom of the group. This leads to the computational advantage that we can estimate the correlations once and then calculate the log-likelihood for each number of degrees of freedom, instead of calculating the correlations again for each copula for all degrees of freedom. Other correlation estimators depend on the marginal distributions, which are in turn influenced by the number of degrees of freedom of the group. Therefore it is necessary to recompute the correlation matrix in every step of the estimation of these copula's parameters.

One disadvantage of Kendall's tau approach is that it provides a correlation matrix ρ which is symmetric, but does not need to be positive definite. To arrive at a positive definite correlations matrix we perform the following transformation of ρ (see i.e. Rousseeuw and Molenberghs [1993]). In this method the correlation matrix ρ is written as $\rho = PDP^T$, where D is the diagonal matrix of the eigenvalues. A positive definite matrix would have positive eigenvalues, therefore in a next step we replace the non-positive eigenvalues by a small positive number (respectively by their absolute value). Then the new matrix ρ^1 is computed using the old eigenvectors and the new modified eigenvalues. By construction the matrix ρ^1 should be positive definite, but this property is not numerically stable. If we rescale ρ^1 by

$$\rho^2_{ij} := \frac{\rho^1_{ij}}{\sqrt{\rho^1_{ii}\rho^1_{jj}}} \tag{A.20}$$

we obtain a unit diagonal. The resulting correlation matrix is positive definite, and this feature is numerically more stable.

Estimation of the copula degrees of freedom

To estimate the copula degrees of freedom we can use the fact that each group taken by itself has a Student copula and therefore we can use the standard approach for estimating the number of degrees of freedom of the Student copula. We use the maximum likelihood estimator on a multivariate distribution with Student copula and the normalised time series of eq. (A.13) as marginals.

More precisely the copula density is a function of the marginals and correlation. In the estimation of the copula degrees of freedom we can ignore the marginals, because they do not depend on the degree of freedom and have only an additive influence on the objective function. For the correlations, which are also influenced by the copula, we use the matrix ρ^2 of equation (A.20). The resulting log density is a function of the degrees of freedom and of the normalised risk factors.

For a given number of degrees of freedom ν the log-density is computed as

$$\begin{aligned} c(\nu) &= \frac{\Gamma((\nu+m)/2)\Gamma(\nu/2)^{m-1}}{\Gamma((\nu+1)/2)^m\sqrt{\det(\rho)}}, \\ t^{-1}(y) &= t^{-1}(u,\nu), \\ c_t(\nu,t) &= \frac{\nu+m}{2}\ln\left(\frac{1+z\rho^{-1}z'}{\nu}\right) - \ln(c(\nu)) - \sum_{j=1}^{m}\left(\frac{\nu+1}{2}ln(1+\frac{t_j^2}{\nu})\right), \\ \ln(f(\nu,u)) &= \sum_{i=1}^{n} c_t(\nu,y_i), \end{aligned}$$

where c_t differs from the log-copula density only by a constant and $\ln(f(\nu,u))$ differs from the log-likelihood function only by a constant which does not depend on the numbers of degrees of freedom. $\ln(f(\nu,u))$ is the objective function. The ν for which it is maximised is our estimate of the copula number of degrees of freedom.

An alternative approach would be to use the full density function, i.e. the density function of the whole vector of risk factors. In this approach we have an m dimensional optimisation problem (instead of m optimisations in 1 dimension). The multivariate density can only be evaluated by numeric integration. Therefore this approach will be much slower.

A.1.4 Drawing Scenarios from Distributions with Grouped t-Copula

In this section we describe ways how to draw scenarios from the unconditional and the conditional distributions with grouped t-copula and arbitrary marginals.

Drawing Scenarios from the Unconditional Distribution

The goal is to draw n-dimensional scenarios from a distribution with marginal distribution functions H_i, $i = 1,\ldots,n$, and grouped t-copula with m groups of size $s_1,\ldots,s_m$ and numbers of degrees of freedom $\nu_1,\ldots,\nu_m$. One can proceed in the following way.

1. *Generate a random vector $Z \sim N(0,\rho)$, where ρ is a linear correlation matrix, and generate an independent random variable $U \sim U(0,1)$*
2. *Denote by G_ν the distribution function of χ^2_ν and let $R_k := G^{-1}_{\nu_k}(U)$ for $k = 1,...,m$.*
3. *Then the vector*

$$Y = \left(\frac{Z_1}{\sqrt{R_1/\nu_1}},\ldots,\frac{Z_{s_1}}{\sqrt{R_1/\nu_1}},\frac{Z_{s_1+1}}{\sqrt{R_2/\nu_2}},\ldots,\frac{Z_{s_1+s_2}}{\sqrt{R_2/\nu_2}},\ldots\frac{Z_n}{\sqrt{R_m/\nu_m}}\right)$$

has by equation (A.18) a grouped t-copula with Student marginals.

4. *Denote by t_ν the distribution function of the one-dimensional Student distribution. Then*

$$X := \left(H_1^{-1}(t_{\nu_1}(Y_1)), \ldots, H_{s_1}^{-1}(t_{\nu_1}(Y_{s_1})), H_{s_1+1}^{-1}(t_{\nu_2}(Y_{s_1+1})), \ldots, H_n^{-1}(t_{\nu_m}(Y_{s_n}))\right),$$

have grouped t-copula and marginal distribution functions H_i, $i = 1, \ldots, n$.

Daul et al. [2004] proves that scenarios constructed in this way have the right distribution. This also follows directly from the proof in Section A.1.4.

Drawing Scenarios from the Conditional Distribution Given the Value of One Component

The goal is to draw n-dimensional scenarios from the distribution which marginal distribution functions H_i, $i = 1, \ldots, n$, and grouped t-copula with m groups of size $s_1, \ldots, s_m$ and numbers of degrees of freedom $\nu_1, \ldots, \nu_m$—conditional on the value of the first component being c: $X_1 = c$.

1. *Generate a random variable $D \sim \chi^2_{\nu_1}$ and set*

$$U := G_{\nu_1}\left(\frac{D}{1 + (t_{\nu_1}^{-1}(H_1(c)))^2/\nu_1}\right)$$

and

$$z_1 := \sqrt{\frac{D}{\nu_1 + (t_{\nu_1}^{-1}(H_1(c)))^2}}\, t_{\nu_1}^{-1}(H_1(c)). \tag{A.21}$$

2. *Generate an n-dimensional random vector $Z \sim (N(0,\rho)|Z_1 = z_1)$ of the multivariate normal conditional on the value of the first component being z_1. Then apply Steps 3 and 4 from Section A.1.4 to get the conditioned scenario from the grouped t-copula.*

To see that the resulting scenarios have the right distribution one can argue as follows. A random vector X with given copula C and marginals F_i can be written as

$$F_X(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Define a vector Y by $Y_i = M_i(X_i)$ for $i = 1, \ldots, n$ and some invertible functions M_i. Then we have

$$\begin{aligned}
F_Y(x_1, \ldots, x_n) &= P(\{Y_1, \ldots, Y_n\} \leq \{x_1, \ldots, x_n\}) \\
&= P(\{X_1, \ldots, X_n\} \leq \{M_1^{-1}(x_1), \ldots, M_n^{-1}(x_n)\} \\
&= F_X(M_1^{-1}(x_1), \ldots, M_n^{-1}(x_n)) \\
&= C(F_1(M_1^{-1}(x_1)), \ldots, F_n(M_n^{-1}(x_n))),
\end{aligned}$$

For $M_i(x) = G_i^{-1}(F_i(x))$ we have

$$F_i(M_i^{-1}(x_i)) = F_i(F_i^{-1}(G_i(x_i)) = G_i(x_i).$$

This implies that Y has copula C and marginals G_i. In this way we can generate from random vectors from a distribution with known marginals and some copula random vectors from another distribution with the same copula and arbitrarily specified marginals.

Now let us turn to generating scenarios from the conditional distribution with grouped t-copula and some specified marginals, conditional on $X_1 = c$. By the definition of Y in Step 3 of Section A.1.4, $X_1 = c$ amounts to $Z_1/\sqrt{R_1/\nu_1} = t_{\nu_1}^{-1}(H_1(c))$, where $Z_1 \sim N(0,1)$ and $R_1 \sim \chi^2_{\nu_1}$. Taking $D := R_1 a$ for some constant, which will be set later, this condition reads $Z_1 = \sqrt{D/(\nu a)}t_{\nu_1}^{-1}(H_1(c))$. Therefore the density function of D is proportional to

$$\begin{aligned} f_D(z) &\sim f_{\chi^2_{\nu_1}}\left(\frac{z}{a}\right) f_{N(0,1)}\left(\sqrt{\frac{z}{\nu a}}t_{\nu_1}^{-1}(H_1(c))\right) \\ &\sim \left(\frac{z}{a}\right)^{(\nu_1-2)/2} \exp\left[-\frac{z/a}{2}\right] \exp\left[-\frac{\frac{z}{\nu a}(t_{\nu_1}^{-1}(H_1(c)))^2}{2}\right] \\ &\sim z^{(\nu_1-2)/2} \exp\left[-\frac{z}{2}\frac{1+(t_{\nu_1}^{-1}(H_1(c)))^2/\nu}{a}\right]. \end{aligned}$$

For $a := 1 + (t_{\nu_1}^{-1}(H_1(c)))^2/\nu$ we get $D \sim \chi^2_{\nu_1}$, which implies that the distributions of R_1 and $Z_1, Z_2, \ldots, Z_n$ are independent from R_1, and therefore can be generated as a normal vector $Z \sim (N(0,\rho)|Z_1 = z_1)$.

Drawing Scenarios from the Conditional Distribution Given the Values of Several Components in the Same Group

The goal is to draw n-dimensional scenarios from the distribution with marginal distribution functions H_i, $i = 1,\ldots,n$, and grouped t-copula with m groups of size $s_{1,\ldots m}$ and numbers of degrees of freedom $\nu_1,\ldots,\nu_m$—conditional on the value of the first k components being $c_1,\ldots,c_k$: $X_1 = c_1, \ldots, X_k = c_k$. Define the k-dimensional vector $\mathbf{d} := (t_{\nu_1}^{-1}(H_1(c_1))/\sqrt{\nu_1},\ldots, t_{\nu_1}^{-1}(H_k(c_k))/\sqrt{\nu_1})$.

1. *Generate a random variable $D \sim \chi^2_{\nu_1}$ and set $U := G_{\nu_1}(D/a)$, where a is $a := 1 + \mathbf{d}\cdot\rho_{k,k}^{-1}\cdot\mathbf{d}^T$ and $\rho_{k,k}^{-1}$ is the correlation submatrix of the first k components. For $i = 1,\ldots,k$ take*

$$z_i := \sqrt{\frac{D}{a}}d_i. \tag{A.22}$$

2. *Generate an n-dimensional random vector $Z \sim (N(0,\rho)|Z_1 = z_1,\ldots, Z_k = z_k)$ of the multivariate normal conditional on the value of the first k components having the values $z_1,\ldots z_k$. Then apply Steps 3 and 4 from Section A.1.4 to get the scenario.*

The proof is similar to the proof in Section A.1.4.

Drawing Scenarios from the Conditional Distribution Given the Values of Several Components across Different Groups

In the next step we will only condition on the vector $\mathbf{Y}$, conditioning on $\mathbf{X}$ can be done analytically. For example we have

$$Y_i = \frac{Z_i}{\sqrt{R_i/\nu_i}} = c_i, \tag{A.23}$$

for $i \in I$, where I denotes the set of components on which the conditions are placed. The most natural parametrization is by a random variable U

$$\begin{aligned} R_i &= G_{\nu_i}^{-1}(U), \\ Z_i &= c_i\sqrt{R_i/\nu_i}, \end{aligned} \tag{A.24}$$

Therefore, if we know the value of U, then we know the distribution of all other variables (all R_i, Z_i for $i \in I$). The non-conditioned variables can be generated in the same way as described in Section A.1.4. Therefore our main goal is to find the conditional distribution of the random variable U, where the condition is given by equation (A.24)

$$P(U < z|Y_i = c_i, i \in I) \sim \int_0^z f_{U(0,1)}(z) f_{N(0,\rho_I)}\left(c_{i_1}\sqrt{G_{\nu_{i_1}}^{-1}(u)/\nu_{i_1}}, \ldots, c_{i_m}\sqrt{G_{\nu_{i_m}}^{-i_m}(u)/\nu_{i_m}}\right) du,$$

where $i_1, \ldots, i_m$ denotes the conditioned components and ρ_I represents the linear correlation matrix of these conditioned components. This equation give us the probability density function of the random variable U under the given condition.

$$f_U(z) \sim f_{N(0,\rho_I)}\left(c_{i_1}\sqrt{G_{\nu_{i_1}}^{-1}(z)/\nu_{i_1}}, \ldots, c_{i_m}\sqrt{G_{\nu_{i_m}}^{-1}(z)/\nu_{i_m}}\right).$$

In general the conditioned distribution of U is not an analytically defined distribution function, but from the density we can numerically approximate the cumulative distribution function. From this we calculate the inverse of the cumulative distribution function. Knowing the inverse of cumulative distribution function we proceed in following way

1. *Estimate the inverse cumulative distribution function $F_U^{-1}(x)$ for variable U.*
2. *Generate a realization from the random variable V with uniform distribution.*
3. *Compute $F_U^{-1}(V)$, which has the desired distribution. With given U compute all R_i, $i = 1, \ldots, n$ and z_i for $i \in I$.*
4. *Generate an n-dimensional random vector $Z \sim (N(0,\rho)|Z_i = z_i, i \in I)$ of the multivariate normal conditional on the value of the components from I having the values $z_i, i \in I$. Then apply Steps 3 and 4 from Section A.1.4 to get the conditioned scenario.*

A.2 Credit Risk Model

A.2.1 From Macroeconomic Shocks to Industry Sector Probabilities of Default

We have given $i = 1, \ldots, I$ industry sectors. At time t industry sector i has a loan default rate $\mu_{i,t}$ given by

$$\mu_{i,t} = \frac{\text{Number of defaults}_{i,t}}{\text{Number of firms}_{i,t}}, \tag{A.25}$$

The loan default rate in industry i depends on a set of macroeconomic risk factors and a noise term via the relation

$$\mu_{i,t} = G(X_t\beta_i) + \varepsilon_{i,t} \tag{A.26}$$

where $G(z)$ is given by

$$G(z) = \Lambda(z) = \frac{e^z}{1+e^z} \tag{A.27}$$

X_t is a $1 \times k$ vector of macroeconomic risk factors, β_i is a $k \times 1$ vector of parameters, and $\varepsilon_{i,t}$ is a noise term from a normal distribution with mean 0 and variance equal to $\hat{\sigma}_i G(\hat{X}_s\hat{\beta}_i)(1 - G(\hat{X}_s\hat{\beta}_i))$. We assume that

$$E(\mu_{i,t}|X_t) = G(X_t\beta_i) \tag{A.28}$$

and $Cov\,(\varepsilon_{i,t}, \varepsilon_{j,l}|X_t, X_l) = 0$ whenever $i \neq j$ or $t \neq l$.

The estimation of parameters is based on Papke and Wooldrige [1996]. Parameters for each sector are estimated separately. The subscript i is dropped in the sequel. The estimation is done using a quasi-likelihood method where the log-likelihood is given by

$$\ln L(b) = \sum_{t=1}^{T} \{\mu_t \ln[G(X_t b)] + (1-\mu_t) ln[1 - G(X_t b)]\} \tag{A.29}$$

where X_t is the t–th row of X. Maximizing Equation A.29 with respect to b gives a consistent estimate of β if Equation A.28 holds. The first order conditions require

$$\frac{\partial lnL}{\partial b} = \sum_{t=1}^{T} \left\{ \frac{\mu_t g(X_t b)}{G(X_t b)} - \frac{(1-\mu_t) g(X_t b)}{1 - G(X_t b)} \right\} X_t = 0 \tag{A.30}$$

where $g(z) = dG(z)/dz$.

In our case it holds that $g(z) = G(z)(1 - G(z)) = \Lambda(z)(1 - \Lambda(z))$. This yields

$$\frac{\partial lnL}{\partial b} = \sum_{t=1}^{T} \{\mu_t - \Lambda(X_t b)\} X_t = 0 \tag{A.31}$$

If we stack μ_t and $\Lambda(X_t b)$ and denote this by μ and Λ we get

$$\frac{\partial lnL}{\partial b} = X'(\mu - \Lambda) = 0 \tag{A.32}$$

Define $\hat{g}_t \equiv g(X_t\hat{b})$ and $\hat{G}_t \equiv G(X_t\hat{b})$ where $\hat{b}$ is the QMLE estimator. The estimated information matrix of the QMLE is then given by

$$\hat{A} = \sum_{t=1}^{T} \frac{\hat{g}_t^2 X_t' X_t}{\hat{G}_t \left(1 - \hat{G}_t\right)} \tag{A.33}$$

In our case this reduces to

$$\hat{A} = \sum_{t=1}^{T} \hat{\Lambda}_t \left(1 - \hat{\Lambda}_t\right) X_t' X_t \tag{A.34}$$

Now define $\hat{D} = diag(\hat{\Lambda})$. Then in matrix notation

$$\hat{A} = X'\hat{D}(I - \hat{D})X \tag{A.35}$$

where I is the T–dimensional identity matrix.

To get a consistent estimator of the true asymptotic standard error of the parameters we also need the outer product of the score. Let $\hat{u}_t \equiv \mu_t - G(X_t\hat{b})$ be the residuals and define

$$\hat{B} = \sum_{t=1}^{T} \frac{\hat{u}_t^2 \hat{g}_t^2 X_t' X_t}{\left[\hat{G}_t(1 - \hat{G}_t)\right]^2} \tag{A.36}$$

Then a valid estimate of the asymptotic variance of $\hat{b}$ is given by

$$\hat{A}^{-1}\hat{B}\hat{A}^{-1} \tag{A.37}$$

Define $\hat{E} \equiv diag(\mu - \hat{\Lambda})$. Then in our case $\hat{B}$ reduces to

$$\hat{B} = X'E^2X \tag{A.38}$$

To select to best model for each industry sector, the likelihood and the BIC criterion are used. The first model selection is done by finding the model with the largest likelihood; the second by finding the model with the smallest BIC.

A.2.2 From Industry Default Probabilities to Loss Distributions

Each bank has a set $\mathcal{I} = \{1, ..., I\}$ of non-interbank obligors. Credit defaults of corporates are described by a Bernoulli random variable X_i where

$$X_i = \begin{cases} 1 & \text{if obligor } i \text{ defaults at time } T \\ 0 & \text{otherwise} \end{cases} \tag{A.39}$$

and $Prob(X_i = 1) = p_i$.

Definition A.1. *Let X be a discrete random variable. The probability generating function of X is defined as*

$$G_X(z) = \sum_{i=0}^{\infty} Prob(X = i)z^i.$$

The probability generation function of a Bernoulli random variable is given by

$$G_{X_i}(z) = 1 - p_i + p_i z = 1 + p_i(z - 1)$$

If default events for the individual obligors $i \in \mathcal{I}$ are independent the probability generating function of default events for the whole portfolio is the product of the individual probability generating functions.

$$G_X(z) = \prod_{i=1}^{N} G_{X_i}(z) = \prod_{i=1}^{N}(1 + p_i(z - 1))$$

Writing this expression in logarithms this gives the expression

$$\log(G(z)) = \sum_{i=1}^{N} \log(1 + p_i(z-1)) \tag{A.40}$$

If the default probabilities are uniformly small, we can ignore terms of degree 2 and higher in the default probabilities. Using a Taylor series expansion of the function $log(1+x)$ at $x_0 = 0$ this gives us:

$$\log(1 + p_i(z-1) \approx p_i(z-1) \tag{A.41}$$

Thus we write equation (A.40) as

$$G(z) = e^{\sum_{i=1}^{N} p_i(z-1)} \tag{A.42}$$

which will hold exactly in the limit, when default probabilities go to zero. Denote the expected number of default events by

$$\mu = \sum_{i=1}^{N} p_i \tag{A.43}$$

To identify the distribution that corresponds to the probability generating function $G(z)$, expand $G(z)$ in its Taylor series, which gives

$$G(z) = e^{\mu(z-1)} = e^{-\mu} e^{\mu z} = \sum_{i=0}^{\infty} \frac{e^{-\mu}\mu^i}{i!} z^i \tag{A.44}$$

This gives an expression for the probability of realizing i defaults over the holding horizon of the loan portfolio under consideration. For individual defaults small, the probability of realizing i default events over the portfolio horizon therefore follows a Poisson distribution with mean μ.

For analyzing the credit losses we want of course to understand the distribution of portfolio losses over the portfolio horizon and not the distribution of default events alone. To reduce the computational effort the exposures in a given loan portfolio are first grouped into exposure bands. Choose an exposure unit U first (for instance Elsinger et al. [2006a] choose the exposure unit U to be 360.000 euro). Denote an obligors expected loss by EL_i and its exposure at default by EAD_i and its loss given default by LGD_i. The exposure that can be lost after an obligor's default is then

$$E_i = EAD_i \times LGD_i$$

The exposure ν_i and the expected loss ϵ_i in multiples of the exposure unit U is given by $\nu_i = E_i/U$ and $\epsilon_i = EL_i/U$. The exact exposures are approximated by rounding the exposure ν_i to the nearest integer multiple of the exposure unit U. Thus every exposure E_i is replaced by the closest integer multiple of the exposure unit U. This partitioning of the exposures into exposure bands m_E gives sufficiently fewer exposures than obligors N and should at the same time be close enough to the original portfolio. A rule of thumb that is given by Bluhm et al. [2003] is that the width of exposure bands should be "small" compared to the average exposure size of the portfolio. Note that in our banking sample it might be useful to apply different exposure units to different banks.

Write $i \in [j]$ whenever i is placed into exposure band j. After this grouping process the portfolio is partitioned into m_E exposure bands such that obligors in a common band $[j]$ have all the common exposure $\nu_{[j]}$ E where $\nu_{[j]} \in \mathbb{N}_0$ is the integer multiple of U representing all obligors i with

$$min\{|\nu_i - n| : n \in \mathbb{N}_0\} = |\nu_i - \nu_{[j]}| \tag{A.45}$$

where $i = 1, ..., I$ and $j = 1, ..., m_E$. When ν_i is an odd-integer multiple of 0.5 we take the convention to round up.

To assign default intensities to given exposure bands we work with the obligors individual default intensity p_i over the analysis horizon of the credit risk model. The expected number of defaults in exposure band $[j]$ is therefore

$$\mu_{[j]} = \sum_{i \in [j]} p_i \tag{A.46}$$

The expected loss in band $[j]$ is then simply the product of the expected number of defaults in band $[j]$ with the band exposure. On page 36 of the technical document of Credit Risk+ an adjustment to compensate for the rounding error is suggested. Credit risk+ suggests an adjustment to the default intensities p_i. Bluhm et al. [2003] for instance suggest to define an adjustment factor for each obligor i by

$$\gamma_i = \frac{E_i}{\nu_{[j]} U} \quad i \in [j], j = 1, ..., m_E \tag{A.47}$$

Assume the individual intensities have been adjusted from p_i to $\gamma_i p_i$. The number of expected default events in the entire portfolio is then

$$\mu = \sum_{j=1}^{m_E} \mu_{[j]} = \sum_{j=1}^{m_E} \frac{\epsilon_{[j]}}{\nu_{[j]}} \tag{A.48}$$

The generating function of the loss random variable L_i is given by

$$G_{L_i}(z) = \sum_{i=0}^{\infty} Prob(L_{[j]} = \nu_{[j]} i) z^{\nu_{[j]} i} \tag{A.49}$$

By independence we can write for the whole portfolio the generating function as

$$G_L(z) = \prod_{j=1}^{m_E} \sum_{i=0}^{\infty} Prob(L_{[j]} = \nu_{[j]} i) z^{\nu_{[j]} i} \tag{A.50}$$

This can be written as

$$G_L(z) = \prod_{j=1}^{m_E} \sum_{i=0}^{\infty} Prob(L_{[j]} = i) z^{\nu_{[j]} i} \tag{A.51}$$

Using the Poisson distribution results for the number of defaults

$$G_L(z) = \prod_{j=1}^{m_E} \sum_{i=0}^{\infty} \frac{e^{-\mu} \mu^i}{i!} z^i z^{\nu_{[j]} i} \tag{A.52}$$

This is equal to

$$G_L(z) = \prod_{j=1}^{m_E} e^{-\mu_{[j]}+\mu_{[j]} z^{\nu_{[j]}}} \tag{A.53}$$

which is equal to

$$G_L(z) = \exp\left(\sum_{j=1}^{m_E} \mu_{[j]}(z^{\nu_{[j]}} - 1)\right) \tag{A.54}$$

Probabilities for losses can be calculated by derivatives of the generating function: The probability of losing an amount of nU is given by

$$Prob(L_{[j]} = n) = \frac{1}{n!}\frac{D^n G_L(z)}{dz^n}|_{z=0} = P_n \tag{A.55}$$

In the technical document of Credit Risk+ it is shown that these probabilities can be calculated recursively by

$$P_n = \sum_{\nu[j]\leq n} \frac{\epsilon_j}{n} P_{n-\nu[j]} \tag{A.56}$$

A.3 Estimation of Interbank Loan Matrix L

Assume that we have, in total, K constraints that include all constraints on row and column sums as well as on the value of particular entries. Let us write these constraints as

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{kij} l_{ij} = b_k \tag{A.57}$$

for $k = 1, ..., K$ and $a_{kij} \in \{0, 1\}$. We seek to find the matrix L that has the least discrepancy to some a priori matrix U with respect to the (generalized) cross entropy measure

$$\mathcal{C}(L, U) = \sum_{i=1}^{N}\sum_{j=1}^{N} l_{ij} \ln(\frac{l_{ij}}{u_{ij}}) \tag{A.58}$$

among all the matrices satisfying (A.57) with the convention that $l_{ij} = 0$ whenever $u_{ij} = 0$ and $0 \ln(\frac{0}{0})$ is defined to be 0.

The constraints for the estimations of the matrix L are not always consistent. For instance the liabilities of all banks in sector k against all banks in sector l do typically not equal the claims of all banks in sector l against all banks in sector k. We deal with this problem by applying a two step procedure.

In a first step we replace an a priori matrix U reflecting only possible links between banks by an a priori matrix V that takes actual exposure levels into account. As there are seven sectors we partition V and U into 49 sub-matrices V^{kl} and U^{kl} which describe the liabilities of the banks in sector k against the banks in sector l and our a priori knowledge. Given the bank balance sheet data we define $u_{ij} = 1$ if bank i belonging to sector k might have liabilities against bank j belonging to sector l and $u_{ij} = 0$ otherwise. The (equality) constraints are that the liabilities of bank i against the sector l equal the row sum of the sub-matrix and that the claims of bank j against the sector k equal the column sum of the sub-matrix, i.e.

$$\sum_{j\in l} v_{ij} = \text{liabilities of bank i against sector l} \tag{A.59}$$

$$\sum_{i \in k} v_{ij} = \text{claims of bank j against sector k} \tag{A.60}$$

For the matrices describing claims and liabilities within a sector (i.e. V^{kk}) which has a central institution we get further constraints. Suppose that bank j^* is the central institution. Then

$$v_{ij^*} = \text{liabilities of bank i against central institution} \tag{A.61}$$

$$v_{j^*i} = \text{claims of bank i against central institution} \tag{A.62}$$

Though these constraints are inconsistent given our data, we use the information to get a revised matrix V which reflects our a priori knowledge better than the initial matrix U. Contrary to U which consists only of zeroes and ones, the entries in V are adjusted to the actual exposure levels.[6]

In a second step we recombine the results of the 49 approximations V^{kl} to get an entire $N \times N$ improved a priori matrix V of inter-bank claims and liabilities. Now we replace the original constraints by just requiring that the sum of *all* (inter-bank) liabilities of each bank equals the row sum of L and the sum of *all* claims of each bank equals the column sum of L.

$$\sum_{j=1}^{N} l_{ij} = \text{liabilities of bank i against all other banks} \tag{A.63}$$

$$\sum_{i=1}^{N} l_{ij} = \text{claims of bank j against all other banks} \tag{A.64}$$

Again we face the problem that the sum of all liabilities does not equal the sum of all claims. By scaling them we enforce consistency.[7] Given these constraints and the prior matrix V we estimate the matrix L.

Finally we can use the information on claims and liabilities with the central bank and with banks abroad. By adding two further nodes and by appending the rows and columns for these nodes to the L matrix, we get a closed (consistent) system of the inter-bank network.

A.4 Network Model

The model is an extension of the model of Eisenberg and Noe [2001]. Consider an economy populated by n nodes. Each of these nodes is a distinct economic entity, say a bank, that participates in a clearing network. Each of these nodes is endowed with an exogenous income $e_i \in \mathbb{R}$.[8] Each bank may have nominal obligations to other nodes in the network. The structure of these liabilities is represented by an $n \times n$ matrix L, where $L_{i,j}$ represents the nominal obligation of node i to node j. As in Eisenberg and Noe [2001] these liabilities are non–negative and the diagonal elements of L are zero for obvious reasons. The nodes may also have liabilities of the same seniority as the liabilities to the other nodes, to creditors outside the

[6] Note that the algorithm that calculates the minimum entropy entries does not converge to a solution if data are inconsistent. Thus to arrive at the approximation V we terminate after 10 iterations immediately after all row constraints are fulfilled.

[7] The remaining claims are added to the vector e. Hence they are assumed to be fulfilled exactly.

[8] Contrary to Eisenberg and Noe [2001] e_i might be negative.

network. We denote these aggregate liabilities by $D_i \geq 0$. On top of this banks may hold shares of other banks. We denote this by the matrix $\Theta \in [0,1]^{n\times n}$ where Θ_{ij} denotes the share that bank i holds in bank j. It has to hold that

$$\sum_{i=1}^{n} \Theta_{ij} \leq 1 \tag{A.65}$$

This specification does allow for the case that a bank is among the shareholders of its own shares ($\Theta_{ii} > 0$). The only restriction I impose on the holdings is that there is no group of banks in which *each* bank is completely owned by other banks in that group, in particular $\Theta_{ii} < 1$. This is summarized in the following

Assumption There exists no subset $\mathcal{I} \subset \{1,\ldots,n\}$ such that

$$\sum_{i\in\mathcal{I}} \Theta_{ij} = 1 \quad \text{for all} \quad j \in \mathcal{I}.$$

Θ is called a *holding matrix* if it fulfills this assumption.

Any node may hold shares of companies outside the network. As the value of these companies is not determined endogenously the value of this holdings is contained in e.

A node is in default whenever the endowment plus the amounts received from other nodes are insufficient to cover the liabilities. In case of default the clearing procedure has to respect three criteria:

1. limited liability, which requires that the total payments made by a node must never exceed the cash flow available to the node,

2. priority of debt claims, which requires that stockholders in the node receive no value unless the node is able to pay off all of its outstanding debt completely, and

3. proportionality, which requires that in case of default all claimant nodes are paid off in proportion to the size of their claim on firm assets.

To operationalize proportionality let $\bar{p}_i$ be the total nominal obligations of node i, i.e.

$$\bar{p}_i = \sum_{j=1}^{n} L_{ij} + D_i \tag{A.66}$$

and define the proportionality matrix Π by

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.67}$$

Evidently, it has to hold that $\Pi \cdot \vec{1} \leq \vec{1}$ where $\vec{1}$ is an $n \times 1$ vector of ones. Let $p = (p_1, \ldots, p_n)' \in \mathbb{R}^n_+$ be the *actual* dollar payments made by banks to its interbank and non interbank creditors under the clearing mechanism. Let $V \geq \vec{0}$ be a vector of equity values and define the map

$$\Upsilon(V, p, e, \Pi, \Theta) = [e + \Pi' p - p + \Theta V] \vee \vec{0} \tag{A.68}$$

For a given vector p a consistent vector of equity values $V^*(p)$ is a fixed point of $\Upsilon(\cdot; p, e, \Pi, \Theta) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$V^*(p) = [e + \Pi' p - p + \Theta V^*(p)] \vee \vec{0} \tag{A.69}$$

where $\vec{0}$ denotes the $n \times 1$ dimensional zero vector. Applying Lemma A.12 in section A.4.4 establishes that there exists a unique fixed point, $V^*(p)$. A vector of actual payments p^* that respects the clearing criteria, i.e.

$$p_i^* = \begin{cases} 0 & \text{if} \quad e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \le 0 \\ e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) & \text{if} \quad 0 \le e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \le \bar{p}_i \\ \bar{p}_i & \text{if} \quad \bar{p}_i \le e_i + \sum\limits_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*)\right) \end{cases}$$

is called a clearing payment vector. This can be summarized as

Definition A.2. *A vector $p^* \in [\vec{0}, \bar{p}]$ is a* **clearing payment vector** *if and only if*

$$p^* = \left\{ [e + \Pi' p^* + \Theta V^*(p^*)] \vee \vec{0} \right\} \wedge \bar{p} \tag{A.70}$$

where $V^*(p^*)$ is the unique solution of Equation A.69.

If p^* is a clearing vector it has to hold that $p^* = \{p^* + V^*(p^*)\} \wedge \bar{p}$ but not vice versa.

A clearing vector p^* is a fixed point of the map $\Phi(\cdot; \Pi, \bar{p}, e, \Theta) : [\vec{0}, \bar{p}] \to [\vec{0}, \bar{p}]$ defined by

$$\Phi(p; \Pi, \bar{p}, e, \Theta) = \left\{ [e + \Pi' p + \Theta V^*(p)] \vee \vec{0} \right\} \wedge \bar{p} \tag{A.71}$$

A.4.1 Existence and Uniqueness of a Clearing Payment Vector

To show existence and uniqueness of a clearing payment vector we choose an indirect route. We introduce an auxiliary problem and show that any solution of the auxiliary problem is also a solution of the original clearing problem and vice versa. In a second step we establish existence and uniqueness of a solution of the auxiliary problem.

The auxiliary problem can be formulated as follows. Let

$$\Omega(W, p, \bar{p}, e, \Pi, \Theta) = e + \Pi' p - \bar{p} + \Theta(W \vee \vec{0}) \tag{A.72}$$

and denote any fixed point of $\Omega(\cdot; p, \bar{p}, e, \Pi, \Theta) : \mathbb{R}^n \to \mathbb{R}^n$ by $W^*(p)$, i.e.

$$W^*(p) = [e + \Pi' p - \bar{p}] + \Theta(W^*(p) \vee \vec{0}) \tag{A.73}$$

and let

$$\Psi(p; \Pi, \bar{p}, e, \Theta) = \left\{ \left[e + \Pi' p + \Theta(W^*(p) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p} \tag{A.74}$$

Call any fixed point p^* of Ψ a solution of the auxiliary problem. The assumption that Θ is a holding matrix guarantees by Lemma A.13 of Appendix A.4.4 that Equation A.73 has a unique solution. Hence, Ψ is well defined.

As a first step we proof that any (super)solution of Φ is a (super) solution of Ψ and vice versa.

Theorem A.3. *Let $\hat{p}$ be a (super)solution of Φ, i.e. $\hat{p} \geq \Phi(\hat{p}; \Pi, \bar{p}, e, \Theta)$. Then $\hat{p}$ is a (super)solution of Ψ with $W^*(\hat{p}) = e + \Pi'\hat{p} - \bar{p} + \Theta V^*(\hat{p})$. If $\hat{p}$ is a (super)solution of Ψ then $\hat{p}$ is a (super)solution of Φ with $V^*(\hat{p}) = (W^*(\hat{p}) \vee \vec{0})$.*

Proof. I prove the assertion for the case of supersolutions. The proof for solutions is completely analogous. Suppose that $\hat{p}$ is a supersolution of Φ, i.e. $\hat{p} \geq \Phi(\hat{p})$. Define $X = e + \Pi'\hat{p} - \bar{p} + \Theta V^*(\hat{p})$. We have to show that X is a solution to Equation A.73, i.e. $X = e + \Pi'\hat{p} - \bar{p} + \Theta(X \vee \vec{0})$. As $\hat{p}$ is a supersolution of Φ, $V_i^*(\hat{p}) > 0$ implies $\hat{p}_i = \bar{p}_i$ and hence $V_i^*(\hat{p}) = X_i$. On the other hand $V^*(\hat{p}) \geq X$. Therefore, $(X \vee \vec{0}) = V(\hat{p})$. This yields

$$X = e + \Pi'\hat{p} - \bar{p} + \Theta(X \vee \vec{0}) \text{ and } \hat{p} \geq \Psi(\hat{p})$$

Now suppose $\hat{p}$ is a supersolution to Ψ. Let $\Lambda = diag(W(\hat{p}) \geq \vec{0})$. Define $X = \Lambda W$. By Equation A.73 it holds that

$$X = \Lambda[e + \Pi'\hat{p} - \bar{p} + \Theta X]$$

As $\hat{p}$ is a supersolution to Ψ it holds that $\Lambda\bar{p} = \Lambda\hat{p}$. Hence,

$$X = \Lambda[e + \Pi'\hat{p} - \hat{p} + \Theta X]$$

It remains to be shown that

$$(I - \Lambda)[e + \Pi'\hat{p} - \hat{p} + \Theta X] \leq \vec{0}$$

As $W_i(\hat{p}) < 0$ implies that

$$\hat{p}_i = max(e_i + \sum_{j=1}^{n} \Pi_{ji}\hat{p}_j + \sum_{j=1}^{n} \Theta_{ij}x_j, 0)$$

it follows that

$$e_i + \sum_{j=1}^{n} \Pi_{ji}\hat{p}_j - \hat{p}_i + \sum_{j=1}^{n} \Theta_{ij}x_j \leq 0$$

whenever $W_i(\hat{p}) < 0$. So X solves A.69. That $\hat{p}$ is a supersolution of Φ is evident. □

If $\hat{p}$ is a supersolution of Ψ then it holds that $V^*(\hat{p}) = [e + \Pi'\hat{p} - \bar{p} + \Theta V^*(\hat{p})] \vee \vec{0}$.

Theorem A.4. *There exists a greatest (p^+) and a least (p^-) clearing vector.*

Proof. If $\Psi(p)$ is a monotone increasing function on the complete lattice $[0, \bar{p}]$ the Tarski Fixed-Point Theorem (see e.g. Zeidler) guarantees that there exists a smallest and a greatest fixed point for the auxiliary problem. Lemma A.16 of Appendix A.4.4 establishes that $W^*(p)$ and therefore $\Psi(p)$ are increasing in p. We know that any solution vector to the auxiliary problem p^* is a solution to the original problem and vice versa. So if there exists a greatest (p^+) and a least (p^-) fixed point for one of the two problems they are the greatest and the least fixed points for the other problem. □

The equity values $V^*(p) = W^*(p) \vee \vec{0}$ of the nodes should not depend on the chosen clearing vector. The next theorem establishes this fact.[9]

Theorem A.5. *The equity values of all nodes are independent of the chosen clearing vector.*

Proof. Let p^* be any clearing vector. It suffices to show that $V^*(p^*) = W^*(p^*)\vee\vec{0} = W^*(p^+) \vee \vec{0} = V^*(p^+)$ where p^+ is the largest clearing vector. We have already established that $W^*(p)$ is increasing in p. Hence, $W^*(p^+) \geq W^*(p^*)$ and therefore $V^*(p^+) \geq V^*(p^*)$. Let $\Lambda = diag(V^*(p^+) > V^*(p^*))$. Note that $\Lambda V^*(p^+) = \Lambda(e + \Pi' p^+ - p^+ + \Theta V^*(p^+))$ and $\Lambda V^*(p^*) \geq \Lambda(e + \Pi' p^* - p^* + \Theta V^*(p^*))$. This implies that

$$\Lambda(V^*(p^+) - V^*(p^*)) \leq \Lambda(\Pi' - \mathbf{I})(p^+ - p^*) + \Lambda\Theta(V^*(p^+) - V^*(p^*))$$

Using

$$V^*(p^+) - V^*(p^*) = \Lambda(V^*(p^+) - V^*(p^*)),$$

rearranging, and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda(V^*(p^+) - V^*(p^*)) \leq \vec{1}'\Lambda(\Pi' - \mathbf{I})(p^+ - p^*) \tag{A.75}$$

Note that the left hand side of Inequality A.75 fulfills the assumptions of Lemma A.10. Hence, if $V^*(p^+) \neq V^*(p^*)$ the left hand side is larger than zero. For the right hand side note that $(p^+ - p^*) \geq \vec{0}$ and $\vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda \leq \vec{0}$. So the right hand side is smaller or equal to 0. Inequality A.75 can only be true if $\Lambda = 0$, i.e. $V^*(p^+) = V^*(p^*)$. ☐

Using the above Theorem we are able to characterize the structure of the system that allows multiple clearing vectors.

Theorem A.6. *Let p^1 and p^2 be two clearing vectors such that $p^1 \geq p^2$. Define $\mathcal{I}$ to be the subset of nodes where p^1 does not equal p^2, i.e. $\mathcal{I} = \{i | p_i^1 > p_i^2\}$. Then it has to hold that $\sum_{j\in\mathcal{I}} \Pi_{ij} = 1$ for all $i \in \mathcal{I}$ and*

$$\sum_{i\in\mathcal{I}} e_i + \sum_{i\in\mathcal{I}}\sum_{j\notin\mathcal{I}} \Pi_{ji} p_j^1 + \sum_{i\in\mathcal{I}}\sum_{j\notin\mathcal{I}} \Theta_{ij} V_j^*(p^1) = 0$$

Proof. Let $\Lambda = diag(p^1 > p^2)$. Note that $p_i^1 > p_i^2 \geq 0$ implies that $\Lambda V^*(p^1) = \Lambda(e + \Pi' p^1 - p^1 + \Theta V^*(p^1))$ and $\Lambda V^*(p^2) \geq \Lambda(e + \Pi' p^2 - p^2 + \Theta V^*(p^2))$. As $V^*(p^1) = V^*(p^2)$ it follows that

$$\vec{0} = \Lambda(V^*(p^1) - V^*(p^2)) \leq \Lambda(\Pi' - \mathbf{I})\Lambda(p^1 - p^2)$$

Summing over both sides yields

$$0 \leq \vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda(p^1 - p^2)$$

This holds only if either $\Lambda = \vec{0}$ or $\vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda = \vec{0}'$. Given the assumption that the clearing vector is not unique and hence $\Lambda \neq \vec{0}$ we get $\sum_{j\in\mathcal{I}} \Pi_{ij} = 1$ for all $i \in \mathcal{I}$.

[9]Eisenberg and Noe [2001] prove this for a model without holdings.

If $p_i^1 > p_i^2$ it has to hold that $V_i^*(p^1) = 0$. Hence,

$$\vec{0} = \Lambda V^*(p^1) = \Lambda(e + \Pi' p^1 - p^1 + \Theta V^*(p^1))$$
$$\Lambda[e + (\Pi' - \mathbf{I})\Lambda p^1 + (\Pi' - \mathbf{I})(\mathbf{I} - \Lambda)p^1 + \Theta(\mathbf{I} - \Lambda)V^*(p^1)]$$

Summation yields

$$0 = \vec{1}'\Lambda[e + (\Pi' - \mathbf{I})\Lambda p^1 + (\Pi' - \mathbf{I})(\mathbf{I} - \Lambda)p^1 + \Theta(\mathbf{I} - \Lambda)V^*(p^1)]$$

Note that $\vec{1}'\Lambda(\Pi' - \mathbf{I})\Lambda p^1 = 0$ and $\vec{1}'\Lambda(\Pi' - \mathbf{I})(\mathbf{I} - \Lambda)p^1 = \vec{1}'\Lambda\Pi'(\mathbf{I} - \Lambda)p^1$. Hence, we get

$$0 = \sum_{i \in \mathcal{I}} e_i + \sum_{i \in \mathcal{I}} \sum_{j \notin \mathcal{I}} \Pi_{ji} p_j^1 + \sum_{i \in \mathcal{I}} \sum_{j \notin \mathcal{I}} \Theta_{ij} V_j^*(p^1)$$

□

The interpretation of the Theorem is straightforward. If there is a subsystem that has no liabilities against banks outside this subsystem and where the sum of all inflows equals zero the clearing vector might be not unique.

A.4.2 Calculating a Clearing Vector

Eisenberg and Noe [2001] interpret e_i as exogenous operating cash flow. They restrict e_i to be non–negative reasoning that any operating costs like wages can be captured by appending a *sink node* to the financial system. Such a sink node has no operating cash flow of its own, nor any obligations to other nodes. The implicit assumption is that the operating costs are of the same priority as the liabilities in the financial system. If these costs are of a higher priority modelling them via a sink node is not possible.[10] Hence, allowing for a more detailed seniority structure makes it necessary not to restrict e_i. Eisenberg and Noe [2001] develop an extremely elegant algorithm called fictitious default algorithm to calculate clearing vectors. Unfortunately, this algorithm brakes down as soon as the restriction that e_i is non–negative is dropped.[11] But it is still possible to define a simple yet less elegant iterative procedure to calculate a clearing vector. Start the algorithm with $p^0 = \bar{p}$ and let $p^{i+1} = [(W^*(p^i) + \bar{p}) \vee \vec{0})] \wedge \bar{p}$.

Lemma A.7. *If Θ is a holding matrix, the sequence $p^{i+1} = [(W^*(p^i) + \bar{p}) \vee \vec{0})] \wedge \bar{p}$ started at $p^0 = \bar{p}$ is well defined, decreasing, and converges to the largest clearing vector p^+.*

Proof. p^{i+1} is well defined if $W^*(p^i)$ is well defined. This is the case as Θ is a holding matrix. $W^*(p^i)$ is calculated applying the procedure in the proof of Lemma A.15. Let $w^0 = e + \Pi' p - \bar{p}$ and $\Lambda^0 = diag(w^0 > \vec{0})$. Calculate $w^k = e + \Pi' p - \bar{p} + \Theta\Lambda^{k-1} w^k$ and $\Lambda^k = diag(w^k > \vec{0})$. This procedure stops after at most n steps and yields $W^*(p^i)$.

To prove that p^i is decreasing note that $p^1 \leq \bar{p} = p^0$ by construction. Now suppose $p^0 \geq p^1 \geq \cdots \geq p^i$. $W^*(p)$ is increasing in p. Hence, $W^*(p^i) \leq W^*(p^{i-1})$ and therefore $p^{i+1} \leq p^i$. Now suppose the series converges to some $\tilde{p}$. This implies

[10] See Section A.4.4 in the Appendix for a simple example.
[11] See Section A.5 in the Appendix for a simple example.

that

$$\tilde{p} = [(W^*(\tilde{p}) + \bar{p}) \vee \vec{0})] \wedge \bar{p} = \left\{ \left[e + \Pi' p + \Theta(W^*(p) \vee \vec{0}) \right] \vee \vec{0} \right\} \wedge \bar{p}$$

So $\tilde{p}$ is a clearing vector. Next note that $W^*(p^0) \geq W^*(p^+)$. This implies that $p^1 \geq p^+$. Now suppose it holds for i up to k that $p^i \geq p^+$. Hence, $W^*(p^k) \geq W^*(p^+)$. But this implies that $p^{k+1} \geq p^+$ and $\tilde{p} \geq p^+$. As p^+ is the largest clearing vector by assumption, $\tilde{p} = p^+$. □

A.4.3 Extensions

Two extensions of the basic model are dealt with in SRM. First it is possible to include bankruptcy costs and second a detailed seniority structure might be included. All the results for the base case remain true for this extensions.

Bankruptcy Costs

Assume that in the case that node i defaults this node faces fixed bankruptcy costs of $b_i > 0$. These costs are deducted from the operating income, i.e. they are of the highest priority. First observe that the definition of a clearing vector has to be adjusted. It has to hold that $p_i^* = \bar{p}_i$ if

$$\bar{p}_i \leq e_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} X_j^*(p^*) \right)$$

or else

$$p_i^* = max(0, e_i - b_i + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} X_j^*(p^*) \right))$$

where $X^*(p)$ is the vector of equity values which remains to be defined. The original definition is not valid anymore. To see this suppose that p^* is a clearing vector and $X(p^*)$ is the corresponding vector of equity values. Now if node i defaults we might get

$$[e_i + \sum_{j=1}^{n} \Pi'_{ji} p_j^* - p_i^* + \sum_{j=1}^{n} \Theta_{ij} X_j(p^*)] \vee 0 \geq 0$$

as

$$p_i^* = [e_i - b_i + \sum_{j=1}^{n} \Pi'_{ji} p_j^* + \sum_{j=1}^{n} \Theta_{ij} X_j(p^*)] \vee 0 = 0$$

A proper definition of equity values in case of bankruptcy costs is

$$V_i^*(p) = e_i - p_i^* + \sum_{j=1}^{n} \left(\Pi_{ji} p_j^* + \Theta_{ij} V_j^*(p^*) \right)$$

if $p_i^* = \bar{p}_i$ or else

$$V_i^*(p) = 0$$

Alternatively we may define the equity value of the nodes using $W^*(p)$ by $X^* = (W^* \vee \vec{0})$.

To calculate a clearing vector in the case of bankruptcy costs, we have to adapt the original procedure slightly. Define $\tilde{W} = W^*(p) - \Lambda b$ where $\Lambda = diag(W^*(p) < \vec{0})$. Let $p^0 = \bar{p}$. And let

$$p^k = \left[\left(\tilde{W}(p^{k-1}) + \bar{p}\right) \vee \vec{0}\right] \wedge \bar{p}$$

Seniority Structure

To adapt the framework to a more elaborate seniority structure I introduce seniority classes. Different liabilities are in the same seniority class if in case of default repayment is rationed proportionally between them. Let $\mathcal{S}_i = (1, 2, \ldots, S_i)$ be a the seniority classes of bank i. Assume that debt claims in class 1 are satisfied first, then the claims in class 2 sequentially up to class S_i are satisfied. Debt claims include interbank positions as well as obligations to parties outside the banking system such as depositors or bondholders. Denote by $\bar{p}_{is} = \sum_{j=1}^{N} l_{ijs} + d_{is}$ the liabilities of bank i in class s. Define

$$\Pi_{ijs} = \begin{cases} \frac{L_{ijs}}{\bar{p}_{is}} & \text{if } \bar{p}_{is} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let Π^s be the matrix consisting of Π_{ijs}, $\bar{p}_{\cdot s} = (\bar{p}_{1s}, \cdots, \bar{p}_{ns})'$ and $S^* = max_i(S_i)$.

Let p_{js} be the actual payments made by bank j in seniority class s. In analogy to the case of just one seniority class a consistent vector of equity values $V^*(p)$ for a given $p = (p_{11} \ldots p_{1S^*}, p_{21} \ldots p_{2S^*}, \ldots, p_{n1} \ldots p_{nS^*})$ is a fixed point of $\Upsilon(\cdot; p, e, \Pi, \Theta) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$V^*(p) = [e + \sum_{s=1}^{S^*} (\Pi^s)' p_{\cdot s} - \sum_{s=1}^{S^*} p_{\cdot s} + \Theta V^*(p)] \vee \vec{0} \tag{A.76}$$

where $p_{\cdot s} = (p_{1s}, \cdots, p_{ns})'$. A clearing payment vector has to satisfy limited liability and absolute priority. **definition** $p^* \geq \vec{0}$ is a clearing vector if and only if $\forall i \in \{1, \ldots, n\}$ and $\forall T \in \{1, \ldots, S^*\}$

$$p^*_{iT} = \min\left(\max\left(e_i + \sum_{j=1}^{N}\sum_{s=1}^{S^*} \Pi_{jis} p^*_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^{N} \Theta_{ij} V^*_j(p^*), 0\right), \bar{p}_{iT}\right)$$

Again a clearing vector can be defined as a fixed point of the map $\Phi(p) = (\Phi_{11} \ldots \Phi_{1S^*}, \Phi_{21} \ldots \Phi_{2S^*}, \ldots, \Phi_{n1} \ldots \Phi_{nS^*}) : [\vec{0}, \bar{p}] \to [\vec{0}, \bar{p}]$ defined by

$$\Phi_{iT} = \left\{\left[e_i + \sum_{j=1}^{N}\sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^{N} \Theta_{ij} V^*_j(p)\right] \vee 0\right\} \wedge \bar{p}_{iT} \tag{A.77}$$

Φ is increasing in p on a complete lattice. Therefore, a greatest (p^+) and a least (p^-) clearing vector exist.

To calculate a clearing vector I use the indirect approach via the auxiliary problem. Let

$$W^*(p) = e + \sum_{s=1}^{S^*} (\Pi^s)' p_{\cdot s} - \sum_{s=1}^{S^*} \bar{p}_{\cdot s} + \Theta(W^*(p) \vee \vec{0}) \tag{A.78}$$

A clearing vector can be characterized by

$$\Phi_{iT} = \left\{ \left[e_i + \sum_{j=1}^{N} \sum_{s=1}^{S^*} \Pi_{jis} p_{js} - \sum_{s=1}^{T-1} \bar{p}_{is} + \sum_{j=1}^{N} \Theta_{ij} (W_j^*(p) \vee 0) \right] \vee 0 \right\} \wedge \bar{p}_{iT} \quad \text{(A.79)}$$

or equivalently

$$\Phi_{iT} = \left\{ \left[W_i(p) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT} \quad \text{(A.80)}$$

The iterattive procedure to calculate a clearing vector is defined as

$$p_{iT}^k = \left\{ \left[W_i(p^{k-1}) + \sum_{s=T}^{S^*} \bar{p}_{is} \right] \vee 0 \right\} \wedge \bar{p}_{iT}$$

with a starting value $p^0 = \bar{p}$.

A.4.4 Shareholding Matrix

Preliminaries

Assumption A.4 guarantees that the equity values are well defined, i.e. $(\mathbf{I} - \Theta)$ is invertible.

Lemma A.8. *Let $\Theta \in [0,1]^{n \times n}$ be the matrix of interbank share holdings and let $\mathbf{I}$ be the $n \times n$ identity matrix. $(\mathbf{I} - \Theta)$ is invertible if an only if Assumption A.4 is satisfied.*

Proof. Assume that there is a subset $\mathcal{I} \subset \{1, \ldots, n\}$ such that $\sum_{i \in \mathcal{I}} \Theta_{ij} = 1$ for all $j \in \mathcal{I}$. Let x be an $n \times 1$ vector with components $x_i = 1$ if $i \in \mathcal{I}$ and $x_i = 0$ otherwise. Clearly $x'(\mathbf{I} - \Theta) = \vec{0}'$. Since $x \neq \vec{0}$, $(\mathbf{I} - \Theta)$ is not injective and thus not invertible.

Now assume that $(\mathbf{I} - \Theta)$ is not invertible. Then there exists a vector $x \neq \vec{0}$ such that $x'(\mathbf{I} - \Theta) = \vec{0}'$. Writing down this system equation by equation we have a linear system given by

$$x_i = \sum_{j=1}^{n} \Theta_{ji} x_j \quad for \quad i = 1, ..., n$$

Taking absolute values on both sides gives

$$|x_i| = |\sum_{j=1}^{n} \Theta_{ji} x_j| \quad for \quad i = 1, ..., n$$

It follows from the triangle inequality that

$$|\sum_{j=1}^{n} \Theta_{ji} x_j| \leq \sum_{j=1}^{n} \Theta_{ji} |x_j| \quad for \quad i = 1, ..., n$$

Now construct an index set $\mathcal{I} \subset \{1, \ldots, n\}$ as follows. A bank i is in $\mathcal{I}$ if and only if $|x_i| \geq |x_j|$ for $j = 1, ..., n$. Since the previous inequality holds for all i it holds in particular for all $i \in \mathcal{I}$. Thus we have

$$|x_i| \leq \sum_{j=1}^{n} \Theta_{ji}|x_j| \leq |x_i| \left(\sum_{j \in \mathcal{I}} \Theta_{ji} + \sum_{j \notin \mathcal{I}} \Theta_{ji} \right) \leq |x_i| \quad \text{for all} \quad i \in \mathcal{I}$$

with equality only if $\sum_{j \in \mathcal{I}} \Theta_{ji} = 1$. Thus invertibility implies that $\sum_{j \in \mathcal{I}} \Theta_{ji} = 1$ for all $i \in \mathcal{I}$ and hence the subset $\mathcal{I}$ violates Assumption A.4. □

Up front it is necessary to introduce a bit of notation.

Definition A.9. *Let y and b be $n \times 1$ vectors. Then $\Lambda = diag(y \geq b)$ is an $n \times n$ diagonal matrix where $\Lambda_{ii} = 1$ if $y_i \geq b_i$ and $\Lambda_{ii} = 0$ otherwise. $diag(y > b)$, $diag(y \leq b)$, $diag(y < b)$, $diag(y \neq b)$, and $diag(y = b)$ are defined analogously.*

Lemma A.10. *Let Θ be an $n \times n$ holding matrix, i.e. Θ fulfills Assumption A.4, and let u be a $n \times 1$ vector. Define $\Lambda = diag(u > \vec{0})$. If $\Lambda \neq \vec{0}$ it holds that $\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda u > 0$.*

Proof. Λ is idempotent. Hence,

$$\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda u = \vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda\Lambda u$$

$\Lambda u \geq 0$ by construction. $\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda \geq \vec{0}'$ as no column sum of Θ exceeds one. This implies that $\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda u \geq 0$. Now, suppose $\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda u = 0$ and define the index set $\mathcal{I} := \{i | u_i > 0\}$. It has to hold that

$$0 = \sum_{i \in \mathcal{I}} u_i - \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \Theta_{ij} u_j = \sum_{i \in \mathcal{I}} u_i - \sum_{j \in \mathcal{I}} \left(\sum_{i \in \mathcal{I}} \Theta_{ij} \right) u_j$$

This implies that $\sum_{i \in \mathcal{I}} \Theta_{ij} = 1$ for all $j \in \mathcal{I}$. But this violates Assumption A.4. □

Taking $x = -u$ yields immediately the following

Corollary A.11. *Let Θ be an $n \times n$ holding matrix, i.e. Θ fulfills Assumption A.4, and let x be a $n \times 1$ vector. Define $\Lambda = diag(x < \vec{0})$. If $\Lambda \neq \vec{0}$ it holds that $\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda x < 0$.*

Equity Values

First I prove that Equation A.69 has a unique solution. Let

$$F(V; u) = [u + \Theta V] \vee 0 \tag{A.81}$$

and define $\tilde{V}$ by

$$\tilde{V} = [u \vee \vec{0}] + \Theta \tilde{V} \tag{A.82}$$

Note that, given Assumption A.4, Lemma A.8 implies that $\tilde{V}$ is well defined and unique. Moreover, $\tilde{V} \geq \vec{0}$ and $F(\tilde{V}; u) \leq \tilde{V}$. To see this consider

$$F(\tilde{V}; u) = [u + \Theta \tilde{V}] \vee 0 = [u - (u \vee \vec{0}) + \tilde{V}] \vee 0 \leq \tilde{V} \tag{A.83}$$

We get the following

Lemma A.12. *Let $u \in \mathbb{R}^n$ and Θ be a holding matrix. Then the map $F(\cdot; u) : \mathbb{R}^n \to \mathbb{R}^n_+$, i.e.*

$$F(V; u) = [u + \Theta V] \vee 0 \tag{A.84}$$

has a unique fixed point, $V^* \geq \vec{0}$.

Proof. If a fixed point V^* exists it has to hold that $V^* \geq \vec{0}$ by construction. Note that $F(V; u)$ is increasing in V. As $F(\vec{0}, u) \geq \vec{0}$, and $F(\tilde{V}; u) \leq \tilde{V}$ the Tarski fixed point theorem implies that there exists a greatest and a smallest fixed point, V^+ and V^-, in the interval $[\vec{0}, \tilde{V}]$. Take any fixed point V^* not necessarily in $[\vec{0}, \tilde{V}]$ and let $\Lambda = diag(V^* > V^-)$. Observe that $\Lambda V^* = \Lambda(u + \Theta V^*)$ and $\Lambda V^- \geq \Lambda(u + \Theta V^-)$. This implies that

$$\Lambda(V^* - V^-) \leq \Lambda\Theta\left(\Lambda(V^* - V^-) + (\mathbf{I} - \Lambda)(V^* - V^-)\right)$$

Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda(V^* - V^-) \leq \vec{1}'\Lambda\Theta(\mathbf{I} - \Lambda)(V^* - V^-)$$

As $(\mathbf{I} - \Lambda)$ is idempotent and $(\mathbf{I} - \Lambda)(V^* - V^-) \leq \vec{0}$ the right hand side of the above inequality is less than or equal to 0. Lemma A.10 implies that the left hand side is larger than 0 as long as $\Lambda \neq \vec{0}$. So it has to hold that $V^* \leq V^-$. As V^- is the smallest fixed point in $[\vec{0}, \tilde{V}]$ it follows that $V^* = V^-$ and the fixed point is unique. □

To prove that Equation A.73 has a unique solution I define

$$G(W; u) = u + \Theta[W \vee 0] \tag{A.85}$$

We get the following

Lemma A.13. *Let $u \in \mathbb{R}^n$ and Θ be a holding matrix. Then the map $G(\cdot; u) : \mathbb{R}^n \to \mathbb{R}^n$, i.e.*

$$G(W; u) = u + \Theta[W \vee 0] \tag{A.86}$$

has a unique fixed point, W^*.

Proof. Let $V^* = F(V^*; u)$ be the unique fixed point of $F(\cdot; u)$ and define $X = u + \Theta V^*$. Let $\Lambda = diag([u + \Theta V^*] \geq \vec{0})$. It holds that $\Lambda X = [X \vee \vec{0}]$ and $V^* = \Lambda(u + \Theta V^*)$. Hence,

$$X = u + \Theta V^* = u + \Theta\Lambda(u + \Theta V^*) = u + \Theta\Lambda X = u + \Theta[X \vee \vec{0}]$$

So X is a solution of A.86.

To prove uniqueness assume there exist two solutions to Equation A.86, W^1 and W^2 and let $\Lambda = diag(W^1 > W^2)$. It holds that

$$\Lambda(W^1 - W^2) = \Lambda\Theta([W^1 \vee 0] - [W^2 \vee 0])$$

Note that $\Lambda([W^1 \vee 0] - [W^2 \vee 0]) \leq \Lambda(W^1 - W^2)$. So we may write

$$\Lambda(W^1 - W^2) \leq \Lambda\Theta(\Lambda(W^1 - W^2) + (\mathbf{I} - \Lambda)([W^1 \vee 0] - [W^2 \vee 0]))$$

Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I}-\Theta)\Lambda(W^1-W^2) \leq \vec{1}'\Lambda\Theta(\mathbf{I}-\Lambda)(W^1-W^2)$$

Again, the right hand side is less than or equal to 0 whereas the left hand side is larger than 0 provided that $\Lambda \neq \vec{0}$. This implies that $W^1 \leq W^2$ and by symmetry $W^2 \leq W^1$. The fixed point is unique. □

Lemma A.13 can be proved constructively, too. Before we are able to this we need the following

Lemma A.14. *Let Ω be a positive $n \times n$ matrix such that no column sum exceeds 1. Let y be a $n \times 1$ vector and let Λ be a matrix of zeros and ones such that $\lambda_{ij} = 0$ for $i \neq j$ and $\lambda_{ii} = 0$ if $y_i < 0$. For any solution of $x = y + \Omega\Lambda x$ it has to hold that $x \geq y$.*

Proof. Let x be such that $x = y + \Omega\Lambda x$ and $U = diag(x < y)$. Premultiply both sides of this equation by U to get $Ux = Uy + U\Omega\Lambda x$. Expand the right hand side to get the equivalent equation $Ux = Uy + U\Omega\Lambda Ux + U\Omega\Lambda(Id-U)x$. By construction $(Id-U)x \geq (Id-U)y$ and $\Lambda(Id-U)y \geq \vec{0}$. Therefore $U\Omega\Lambda(Id-U)y \geq \vec{0}$. From these observations it follows that:

$$\begin{aligned}
Ux - U\Omega\Lambda Ux &= Uy + U\Omega\Lambda(Id-U)x \\
&\geq Uy + U\Omega\Lambda(Id-U)y \\
&\geq Uy
\end{aligned}$$

Now $\vec{1}'(Id - U\Omega\Lambda) \leq \vec{1}$. This implies that $\vec{1}'(Id - U\Omega\Lambda)Ux \leq \vec{1}'Ux$. As long as $U \neq 0$ it holds that $Uy \geq Ux$ and $Uy \neq Ux$. Hence for $U \neq 0$ it holds that $\vec{1}'Uy > \vec{1}'Ux \geq \mathbf{1}(Id - U\Omega\Lambda)Ux \geq \vec{1}Uy$. From this contradiction we conclude that $U = 0$ and $x \geq y$. □

We are now ready to prove Lemma A.13.

Lemma A.15. *Let Θ be an $n \times n$ matrix of shareholdings that fulfills Assumption A.4. Then the equation*

$$W = u + \Theta(W \vee \vec{0})$$

has a unique solution W^* for any $n \times 1$ vector u.

Proof. **Existence**: Let $W^0 = u$ and let $\Lambda^0 = diag(u > \vec{0})$. By Lemma A.8 we know that $W = u + \Theta\Lambda^0 W$ has a unique solution W^1. By Lemma A.14 $W^1 \geq u$. Let $\Lambda^k = diag(W^k > \vec{0})$. The equation $W = u + \Theta\Lambda^k W$ has a unique solution W^{k+1} by Lemma A.8. By construction $\Lambda^k W^k \geq \Lambda^{k-1}W^k$. Therefore we have $u + \Theta\Lambda^k W^k \geq u + \Theta\Lambda^{k-1}W^k = W^k$. Define $y = u + \Theta\Lambda^k W^k - W^k$. It holds that $W^{k+1} - W^k = y + \Theta\Lambda^k(W^{k+1} - W^k)$. Given that $y \geq \vec{0}$, the column sums of Θ do not exceed 1, and $\lambda^k_{ii} < 0$ whenever $y_i < 0$ Lemma A.14 implies that any solution of $x = y + \Theta\Lambda^k x$ has the property that $x \geq y$. In our case this means that $W^{k+1} - W^k \geq y \geq \vec{0}$. This in turn implies that $\Lambda^{k+1} \geq \Lambda^k$. If $\Lambda^k = \Lambda^{k-1}$ it follows that $W^{k+1} = W^k$ and therefore $\Lambda^k W^k = W^k \vee \vec{0}$. Therefore W^k is a solution to $W = u + \Theta(W \vee \vec{0})$. If $\Lambda^k \neq \Lambda^{k-1}$ continue the procedure. The iteration stops after finitely many steps as $\Lambda^k \leq Id$ (at the most n steps).

Uniqueness: To prove *uniqueness* we show first that if W^a and W^b are solutions then there has to exist a $\bar{W} \geq W^* = max(W^a, W^b)$ such that $\bar{W} = u + \Theta(\bar{W} \vee \vec{0})$. Finally we show that this implies that $W^a = W^b = \bar{W}$.

Let $\Lambda^a = diag(W^a \geq \vec{0})$, $\Lambda^b = diag(W^b \geq \vec{0})$, and $\Lambda^* = max(\Lambda^a, \Lambda^b)$. By Lemma A.8 the equation $W = u + \Theta\Lambda^* W$ has a unique solution $\hat{W}$. Let $y = u + \Theta\Lambda^* W^* - W^*$. Evidently, $\Lambda^* W^* \geq max(\Lambda^a W^a, \Lambda^b W^b)$ and hence

$$u + \Theta\Lambda^* W^* \geq max(W^a, W^b) = W^*$$

It follows that $y \geq \vec{0}$. Now $\hat{W} - W^* = y + \Theta\Lambda^*(\hat{W} - W^*)$. Applying Lemma A.14 yields that $\hat{W} \geq W^*$. Define $W^0 := \hat{W}$ and $\Lambda^0 = diag(W^0)$. Using the same iterative procedure as in existence part of the proof we get the result that there exists a $\bar{W}$ such that $\bar{W} = u + \Theta(\bar{W} \vee \vec{0}) \geq W^*$.

Note that

$$\bar{W} - W^a = \Theta\left(\bar{\Lambda}\bar{W} - \Lambda^a W^a\right)$$

Where $\bar{\Lambda} = diag(\bar{W} \geq \vec{0})$. The fact that $\bar{\Lambda} W^a \leq \Lambda^a W^a$ implies that

$$\bar{\Lambda}\Theta\left(\bar{\Lambda}\bar{W} - \Lambda^a W^a\right) = \bar{\Lambda}\left(\bar{W} - W^a\right) \geq \bar{\Lambda}\bar{W} - \Lambda^a W^a.$$

Let $x := \bar{\Lambda}\bar{W} - \Lambda^a W^a$. It is easy to verify that $x \geq 0$. Define $U = diag(x > \vec{0})$ and $\mathcal{I} = \{i | x_i > 0\}$. If $U = \vec{0}$ then $\bar{W} = W^a$. As $Ux = \bar{\Lambda}x = x$ and $U\bar{\Lambda} = U$ it follows that $U\Theta Ux \geq Ux$. Let $\vec{1}$ be an $n \times 1$ vector of ones and suppose that $U \neq 0$. Then $\vec{1}U\Theta U \leq \vec{1}'U$. For all $i \notin \mathcal{I}$ it holds that $[\vec{1}'U\Theta U]_i = [\vec{1}'U]_i = 0$. So if $\vec{1}'U\Theta U \neq \vec{1}'U$ this would imply that the left hand side is smaller than the right hand side for some $i \in \mathcal{I}$. But this in turn would imply that $\vec{1}U\Theta Ux < \vec{1}Ux$. This would be a contradiction to $U\Theta Ux \geq Ux$ and $x \geq 0$. Therefore $\vec{1}'U\Theta U$ has to equal $\vec{1}'U$. This equality holds if and only if

$$\sum_{j \in \mathcal{I}} \theta_{ij} = 1 \quad \forall i \in \mathcal{I}$$

where $\mathcal{I} = \{i | x_i > 0\}$. This is a violation of Assumption A.4 about Θ. Hence, $U = 0$. The solution to our problem is therefore unique. □

Properties of W^*

Theorem A.16. *The unique fixed point $W^*(u)$ of $G(W; u)$ has the following properties*

1. *$W^*(u)$ is increasing in u*
2. *$u^2 \geq u^1$ implies $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$*
3. $\frac{\partial W_i^*}{\partial u_i} \geq 1$
4. $\frac{\partial W_i^*}{\partial u_j} \geq 0$

Proof. First, I show that $W^*(u)$ is increasing in u. Assume u^1 and u^2 are two vectors in $\mathbb{R}^n$ such that $u^2 \geq u^1$. Let $w^1 = u^1 + \Theta max(w^1, 0)$ and $w^2 = u^2 + \Theta max(w^2, 0)$ be the respective fixed points. Let $x = u^2 - u^1 \geq \vec{0}$. It holds that

$$w^2 - w^1 = x + \Theta(max(w^2, 0) - max(w^1, 0))$$

Let $\Lambda = diag(w^1 > w^2)$. Note that $\Lambda(max(w^2,0) - max(w^1,0)) \geq \Lambda(w^2 - w^1)$ and $(I-\Lambda)(max(w^2,0) - max(w^1,0)) \geq 0$. Hence,

$$\begin{aligned}\Lambda(w^2 - w^1) &= \Lambda x + \Lambda\Theta(max(w^2,0) - max(w^1,0)) \\ &= \Lambda x + \Lambda\Theta\Lambda(max(w^2,0) - max(w^1,0)) \\ &+ \Lambda\Theta(I-\Lambda)(max(w^2,0) - max(w^1,0)) \\ &\geq \Lambda x + \Lambda\Theta\Lambda(w^2 - w^1)\end{aligned}$$

Rearranging and premultiplying by $\vec{1}'$ yields

$$\vec{1}'\Lambda(\mathbf{I} - \Theta)\Lambda(w^2 - w^1) \geq \vec{1}'\Lambda x$$

The right hand side is larger or equal to 0. The left hand side is smaller than 0 by Lemma A.10 unless $\Lambda \neq 0$. Hence, $w^2 \geq w^1$.

To prove the second claim note that $W^*(u^2) \geq W^*(u^1)$ implies

$$max(W^*(u^2),0) - max(W^*(u^1),0) \geq 0$$

and hence $W^*(u^2) - W^*(u^1) \geq u^2 - u^1$. This in turn implies that $\frac{\partial W_i^*}{\partial u_i} \geq 1$ and $\frac{\partial W_i^*}{\partial u_j} \geq 0$. □

Sink Nodes

Assume that the financial system consist of two banks. Bank 1 has an operating cash flow of 0.5. Bank 2 has revenues of 2 but has to pay wages of 4. In the interbank market bank 1 ows 1 bank 2 and vice versa. If wages have the same priority as the interbank liabilities we append an additional node 3 to the system for the workers. So

$$e = \begin{pmatrix} \frac{1}{2} \\ 2 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

Clearing the system yields

$$p^* = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad \Pi' p^* + e - p^* = \begin{pmatrix} \frac{1}{10} \\ 0 \\ \frac{24}{10} \end{pmatrix}$$

The shortfall of node 2 is proportionally shared between bank 1 and the workers. The workers lose 1.6 and node 1 loses 0.4. If we assume by contrast that wages are of a higher priority the sink node approach can not be used. Yet, the problem is still well defined and can be solved. The system

$$e = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has the solution

$$p^* = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad \Pi' p^* + e - p^* = \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}$$

In this case node 1 is bankrupt and loses 0.5. The workers lose 1.5. To introduce sink nodes is not as innocuous as it may seem.

A.5 The Fictitious Default Algorithm

To calculate a clearing vector Eisenberg and Noe [2001] propose the following iterative procedure. Let $\Lambda(p) = diag(\Pi' p + e < \bar{p})$ and define the map $p \rightarrow FF'_p(p)$ as follows:

$$FF'_p(p) \equiv \Lambda(p')(\Pi'(\Lambda(p')p + (\mathbf{I} - \Lambda(p'))\bar{p} + e) + (\mathbf{I} - \Lambda(p'))\bar{p}$$

This map returns for all nodes not defaulting under p' the required payment $\bar{p}$. For all other nodes it returns the node's value assuming that nondefaulting nodes pay $\bar{p}$ and defaulting nodes pay p. This map has a unique fixed point which is denoted by $f(p')$. Note that the equation for the fixed point

$$f(p') = \Lambda(p')(\Pi'(\Lambda(p')f(p') + (\mathbf{I} - \Lambda(p'))\bar{p} + e) + (\mathbf{I} - \Lambda(p'))\bar{p}$$

can actually be written quite compactly:

$$(\mathbf{I} - \Lambda(p')\Pi'\Lambda(p'))(f(p') - \bar{p}) = \Lambda(p')(e + \Pi'\bar{p} - \bar{p})$$

Premultiplying this with$\Lambda(p')$ yields

$$\Lambda(p')(\mathbf{I} - \Pi')\Lambda(p')(f(p') - \bar{p}) = \Lambda(p')(e + \Pi'\bar{p} - \bar{p})$$

Hence, to calculate the fixed point it suffices to consider the subsystem of defaulting nodes. This is crucial if the number of nodes is large and default is rare.

Eisenberg and Noe [2001] show that under the assumption that $e \geq \vec{0}$ (and $\Theta = \vec{0}$) the sequence of payment vectors $p^0 = \bar{p}$, $p^i = f(p^{i-1})$ decreases to a clearing vector in at most n interations. The assumption that $e \geq \vec{0}$ is essential as is illustrated by the following example.

$$e = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Setting $p^0 = \bar{p}$ yields

$$\Lambda(p^0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad p^1 = f(p^0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence, p^1 is not a supersolution and the algorithm brakes down.

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